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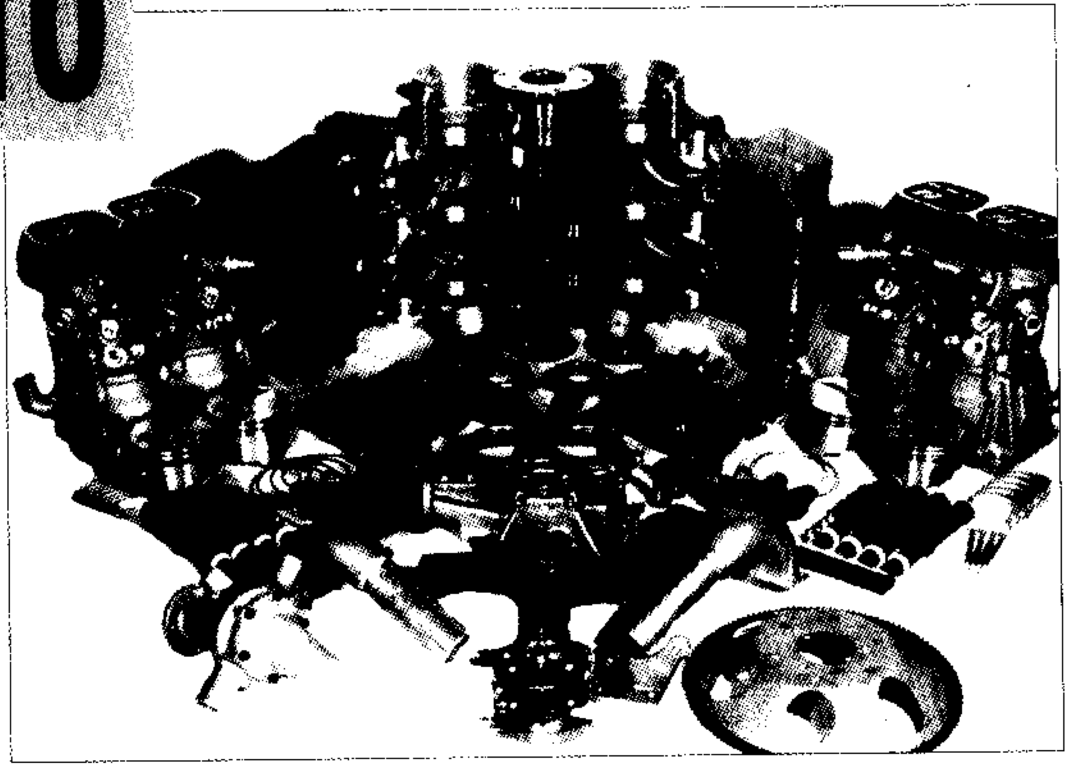


BALANCING

10.1 INTRODUCTION

Unbalance in machine components arises either due to eccentric rotating or reciprocating masses. These masses introduce severe stresses and result in undesirable vibrations in the machines. In this chapter, we shall study the various methods to balance the rotating and reciprocating masses.

The rotating masses may be either in a single plane or in different planes. The reciprocating masses give rise to primary forces and couples and secondary forces and couples. There could be unbalance due to the combined effect of rotating and reciprocating masses.

By balancing we mean to eliminate either partially or completely the effects due to resultant inertia forces and couples to avoid vibration of a machine or device.

10.2 BALANCING OF ROTATING MASSES

10.2.1 Single Rotating Mass

Balance mass in the same plane as the disturbing mass Consider a single mass M rotating with angular speed ω at a radius r, as shown in Fig.10.1. The centrifugal force due to this mass is

$$F_m = Mr\omega^2$$

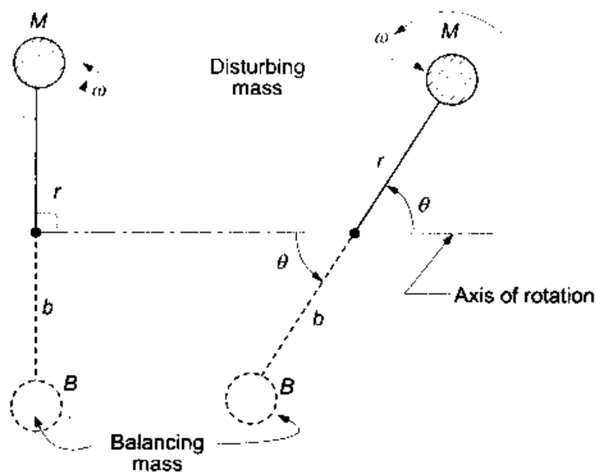


Fig.10.1 Balancing of a single rotating mass by a single mass in the same plane

If a balancing mass B is placed on this rotating machine component in the same plane at a radius b and in line with the mass M at 180°, then the centrifugal force due to mass B will be

$$F_b = Bb\omega^2$$

For the equilibrium of the system, we have

$$F_m = F_b$$

or

$$Mr = Bb \qquad (10.1)$$

Two balance masses in different planes If the balance mass cannot be placed in the same plane as the rotating mass then two parallel masses may be used to balance the rotating mass. The balance masses may be either on the same side of the unbalance mass or on opposite sides. The equilibrium equations would require that the resultant sum of their moments about any point in the same plane must be zero.

1. Balance masses on the same side of the disturbing mass

Consider a mass M at A rotating at a radius r and two balance masses B_1 and B_2 at B and C, parallel to M, rotating at radii b_1 and b_2 respectively, as shown in Fig.10.2(a). Let l_1 and l_2 be the distances of these masses from M.

Taking moments about B, we have

$$Mrl_1 = B_2b_2(l_2 - l_1)$$

or

$$B_2b_2 = Mr\left[\frac{l_1}{l_2 - l_1}\right] \qquad (10.2a)$$

Taking moments about C, we have

$$Mrl_2 = B_1b_1(l_2 - l_1)$$

or

$$B_1b_1 = Mr\left[\frac{l_2}{l_2 - l_1}\right] \qquad (10.2b)$$

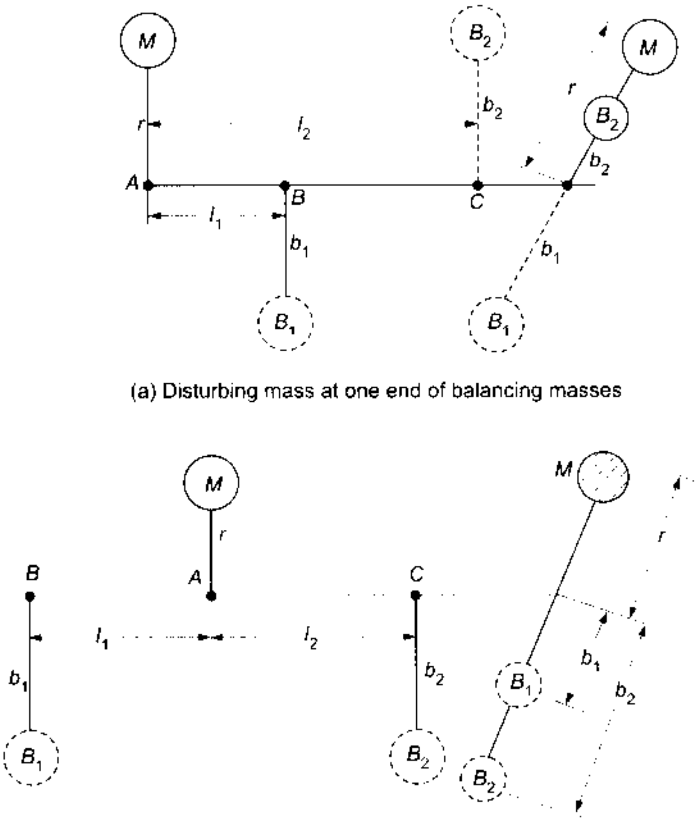
2. Balance masses on the opposite sides of the disturbing mass

Consider the two balance masses B_1 and B_2 on the opposite sides of the disturbing mass M, as shown in Fig.10.2(b). Taking moments about B, we have

$$Mrl_1 = B_2b_2(l_1 + l_2)$$

or

$$B_2b_2 = Mr\left[\frac{l_1}{l_1 + l_2}\right] \quad (10.3a)$$



(a) Disturbing mass at one end of balancing masses

(b) Disturbing mass in between the balancing masses

Fig.10.2 Balancing of a single mass by two masses

Now taking moments about C, we have

$$Mrl_2 = B_1b_1(l_1 + l_2)$$

or

$$B_1b_1 = Mr\left[\frac{l_2}{l_1 + l_2}\right] \quad (10.3b)$$

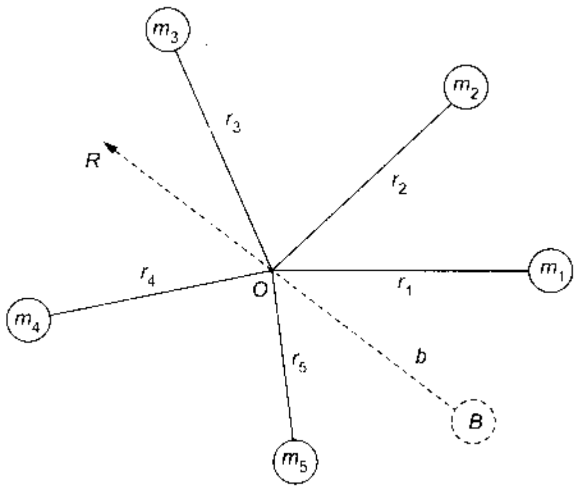
10.2.2 Many Masses Rotating in the Same Plane

Let there be M_i, $i = 1$ to n, masses rotating in the same plane with radii r_i, $i = 1$ to n and with same angular speed ω, as shown in Fig.10.3(a), so that the centrifugal force due to each mass is,

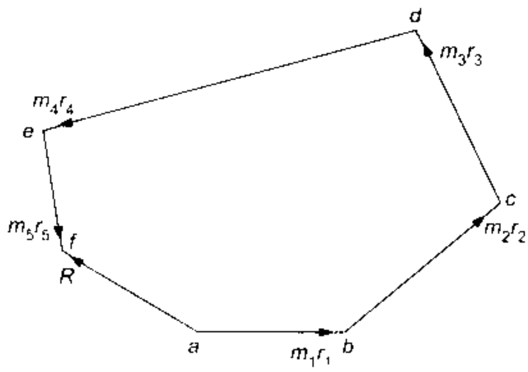
$$F_i = M_i r_i \omega^2$$

Since these forces are in the same plane, therefore, they can be represented by the sides of a regular polygon taken in order, as shown in Fig.10.3(b). Let R be the resultant of these forces. Then the resultant centrifugal force due to R,

$$R = \sum M_i r_i \omega^2$$



(a) Space diagram



(b) Vector diagram

Fig.10.3 Several masses rotating in same plane

If a balancing mass B is placed at a radius b at 180° with R, then the centrifugal force due to B is,

$$F_b = Bb\omega^2$$

For the equilibrium of the system, we have

$$R = F_b$$

or
$$\sum M_i r_i = Bb \tag{10.4}$$

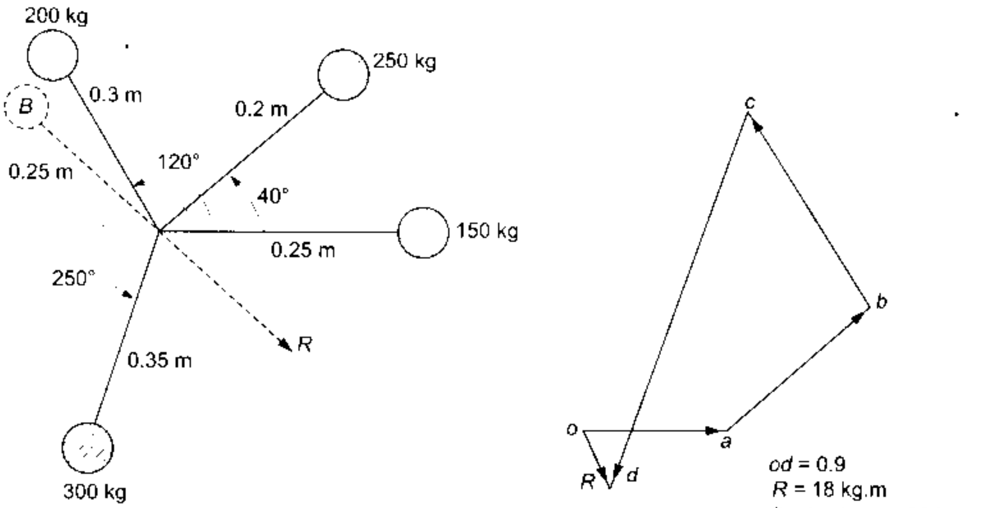
From (10.4), it may be seen that the force polygon may be drawn for $M_i r_i$ instead of $M_i r_i \omega^2$

Example 10.1

Four masses 150 kg, 250 kg, 200 kg and 300 kg are rotating in the same plane at radii of 0.25 m, 0.2 m, 0.3 m, and 0.35 m respectively. Their angular location is 40°, 120°, and 250° from mass 150 kg, respectively measured in counter-clockwise direction. Find the position and magnitude of the balance mass required, if its radius of rotation is 0.25 m.

■ **Solution**

The mass space diagram is shown in Fig.10.4(a). The problem can be solved either analytically or graphically.



(a) Space diagram (b) Vector diagram

Fig.10.4

Analytical method

Table 10.1

M kg	r m	Mr kgm	θ deg	$H = Mr\cos\theta$ kgm	$V = Mr\sin\theta$ kgm
150	0.25	37.5	0	37.5	0
250	0.20	50.0	40	38.3	32.14
200	0.30	60.0	120	−30.0	51.96
300	0.35	105.0	250	−35.9	−98.67

From Table 10.1,

$$\sum H = 9.9 \qquad \sum V = -14.57$$

Resultant,

$$R = \left[\left(\sum H\right)^2 + \left(\sum V\right)^2\right]^{0.5}$$

$$= \left[(9.9)^2 + (-14.57)^2\right]^{0.5} = 17.61 \text{ kgm}$$

Let B be the balancing mass, then

$$0.25B = 17.61$$

or

$$B = 70.46 \text{ kg}$$

Let θ_r be the angle of the resultant with 150 kg mass, then

$$\tan\theta_r = -\frac{-14.57}{-9.9} = -1.47172$$

or

$$\theta_r = -55.8°$$

Angle of the balance mass from the horizontal mass 150 kg is

$$\theta_b = 180° - 55.8° = 124.2° \text{ ccw}$$

Graphical method

The graphical construction is shown in Fig.10.4(b). By measurement:

$R = 17.61$ kgm Then $B = 70.46$ kg, $\theta_r = -55.8°$ and $\theta_b = 124.2°$

10.2.3 Many Masses Rotating in Different Planes

Consider a force F in plane B, as shown in Fig.10.5. Let this force be transferred to a reference plane A at a distance a. The effect of transferring a force F from plane B to plane A is:

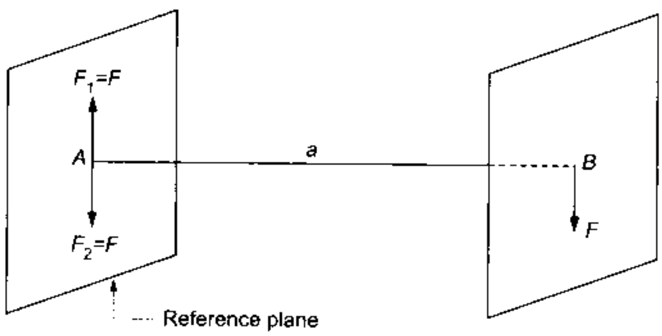


Fig.10.5 Equivalence of a force

1. an unbalance force $F_2 = F$ on plane A, and
2. an unbalanced couple, $C = Fa$.

The couple is represented by a vector at right angles to the plane of the couple and the arrow head points in the direction in which a right hand screw would move if acted upon by the couple. In practice the phase of the couple diagram is rotated through 90° counter-clockwise.

This leads to the balancing equations, in general

$$\sum Mr = 0$$

$$\sum Mra = 0$$

Let us consider the mass system as shown in Fig.10.6(a). The orientation of the forces is shown in Fig.10.6(b). The couples acting on the system are:

$$C_1 = -M_1 r_1 l_1$$
$$C_2 = M_2 r_2 l_2$$
$$C_3 = M_3 r_3 l_3$$
$$C_4 = M_4 r_4 l_4$$

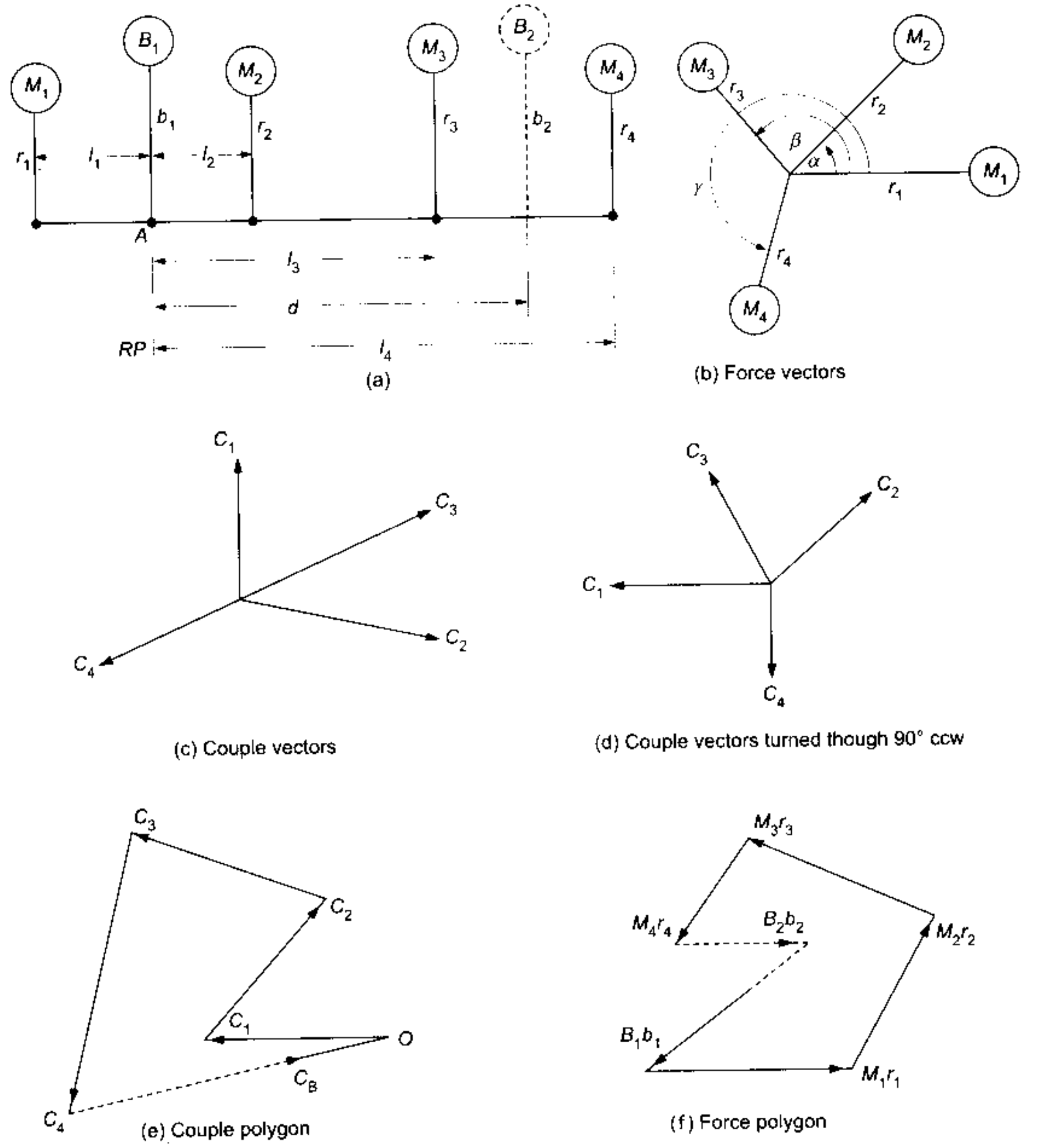


Fig.10.6 Several masses rotating in different planes

The couples are shown in Fig.10.6(c), and when turned through 90°, are shown in Fig.10.6(d). The couple vectors may be fixed in their correct relative positions by drawing them radially outwards along the corresponding radii for all masses which lie on one side of the fixed point *A* and radially inwards along the corresponding radii for all masses which lie on the other side of the fixed point *A*. The fixed point *A* is taken as the point of intersection of the plane of rotation of one of the balancing masses B_1 and the axis of rotation, in order to eliminate the couple due to the mass in this plane. The plane at A is known as the *reference plane*. The couple polygon has been drawn in Fig.10.6(e). The resultant couple is represented by C_B, the closing side of the couple polygon, as shown.

Now $$C_B = B_2 b_2 d$$

or $$B_2 = \frac{C_B}{b_2 d} \qquad (10.5)$$

Knowing B_2 and its direction, draw the force polygon, as shown in Fig.10.6(f). The closing side of the force polygon will represent the magnitude and direction of the force due to the balancing mass required in plane A. The whole process can be represented by the following table:

Table 10.2

Plane	Mass M (1)	Radius r (2)	Mr (3) = (1) × (2)	Distance from plane A l (4)	Couple Mrl (5) = (3) × (4)
1	M_1	r_1	M_1r_1	$-l_1$	$-M_1r_1l_1$
A	B_1	b_1	B_1b_1	0	0
2	M_2	r_2	M_2r_2	l_2	$M_2r_2l_2$
3	M_3	r_3	M_3r_3	l_3	$M_3r_3l_3$
B	B_2	b_2	B_2b_2	d	B_2b_2d
4	M_4	r_4	M_4r_4	l_4	$M_4r_4l_4$

10.2.4 Analytical Method

Several masses in the same plane

Let M_i = number of masses, $i = 1, 2, 3, \cdots$

r_i = radius of mass M_i

θ_i = angle of mass M_i with x-axis measured counter-clockwise

B = balancing mass

b = radius of balancing mass

θ_b = angle of mass B with x-axis measured counter-clockwise

Considering forces along the x and y-axes, we have

$$\sum M_i r_i \cos\theta_i + Bb\cos\theta_b = 0; \qquad \sum M_i r_i \sin\theta_i + Bb\sin\theta_b = 0$$

$$\left[\left(\sum M_i r_i \cos\theta_i\right)^2 + \left(\sum M_i r_i \sin\theta_i\right)^2\right]^{0.5} = Bb$$

or $$B = \frac{\left[\left(\sum M_i r_i \cos\theta_i\right)^2 + \left(\sum M_i r_i \sin\theta_i\right)^2\right]^{0.5}}{b} \qquad (10.6)$$

$$\tan\theta_b = \frac{-\sum M_i r_i \sin\theta_i}{-\sum M_i r_i \cos\theta_i} \tag{10.7}$$

Several masses in different planes If M_L and M_M be the balance forces at radii r_L and r_M respectively, then for the balance of couples about plane L, we have

$$\left[\left(\sum M_i r_i l_i \cos\theta_i\right)^2 + \left(\sum M_i r_i l_i \sin\theta_i\right)^2\right]^{0.5} = M_M r_M l_M \tag{10.8}$$

$$\tan\theta_M = \frac{-\sum M_i r_i l_i \sin\theta_i}{-\sum M_i r_i l_i \cos\theta_i} \tag{10.9}$$

For the balance of forces, we have

$$\left[\left(\sum M_i r_i \cos\theta_i\right)^2 + \left(\sum M_i r_i \sin\theta_i\right)^2\right]^{0.5} = \left[(M_L r_L \cos\theta_L)^2 + (M_M r_M \sin\theta_M)^2\right]^{0.5}$$

$$M_L r_L = \left[\left(\sum M_i r_i \cos\theta_i + M_M r_M \cos\theta_M\right)^2 + \left(\sum M_i r_i \sin\theta_i + M_M r_M \sin\theta_M\right)^2\right]^{0.5} \tag{10.10}$$

$$\tan\theta_M = \frac{-\left(\sum M_i r_i \sin\theta_i + M_M r_M \sin\theta_M\right)}{-\left(\sum M_i r_i \cos\theta_i + M_M r_M \cos\theta_M\right)} \tag{10.11}$$

Example 10.2

A shaft carries four masses as shown in Fig.10.7(a) and (b). The balancing masses are to be placed in planes L and M. If the balancing masses revolve at a radius of 100 mm, find their magnitude and *angular positions*.

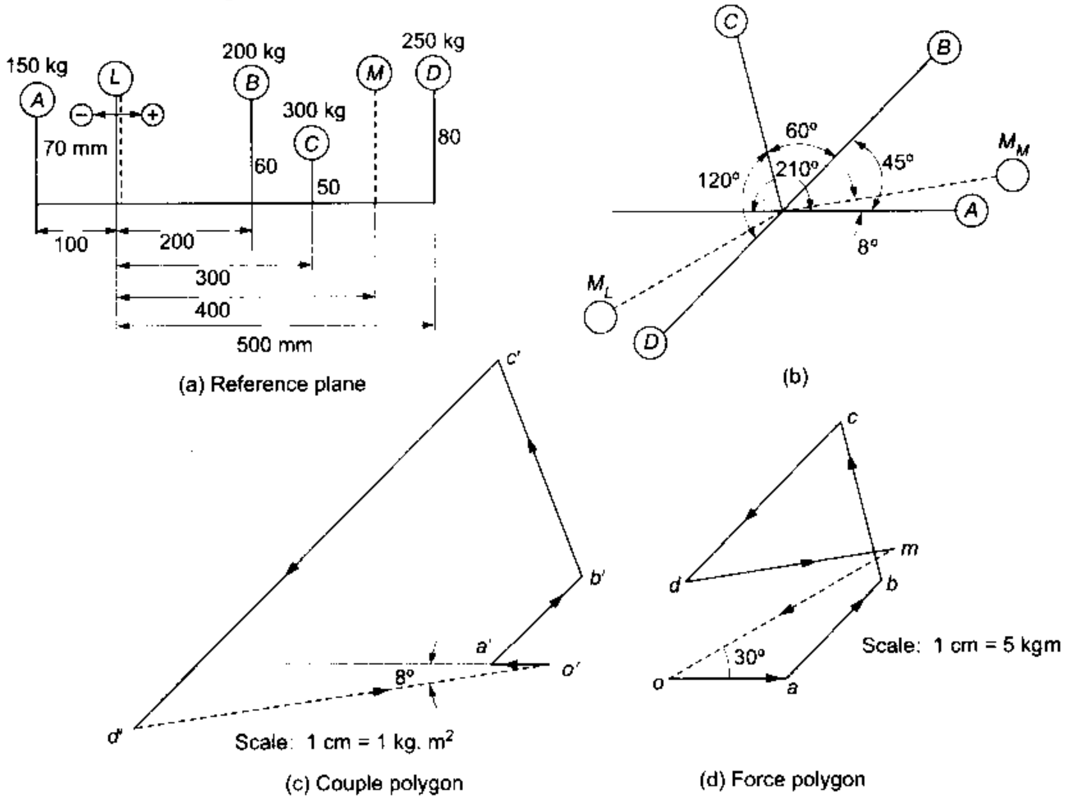


Fig.10.7 Shaft carrying four masses

■ **Solution**

Assume the plane L as the reference plane.

Table 10.3

Plane	Mass, M kg	Radius, r m	Mr kgm	Distance from plane L, l m	Mrl kgm^2
A	150	0.07	10.5	–0.1	–1.05
L	M_L	0.10	0.1 M_L	0	0
B	200	0.06	12.0	0.2	2.40
C	300	0.05	15.0	0.3	4.50
M	M_M	0.10	0.1 M_M	0.4	0.04 M_M
D	250	0.08	20.0	0.5	10.0

1. Draw the couple polygon from the data in the last column of Table 10.3, as shown in Fig.10.7(c). By measurement,

$$0.04M_M = \text{vector } d'o' = 7.7 \text{ cm}$$

or

$$M_M = 192.5 \text{ kg}$$

The angular position of M_M is obtained by drawing *OB* parallel to *d'o'* in Fig.10.7(b). $\theta_M = 8°$.

2. Now draw the force polygon from the data in column 4 of the table, as shown in Fig.10.7(d). The vector *mo* represents the balance force. By measurement

$$0.1M_L = \text{vector } mo = 4.7 \times 5$$

or

$$M_L = 235 \text{ kg}$$

The angular position of M_L is obtained by drawing a line parallel to *mo* in Fig.10.7(b). $\theta_L = 30° + 180° = 210°$.

Analytical method

Reference plane L

From Table 10.4,

$$\sum M_i r_i \cos\theta_i = 0.961$$

$$\sum M_i r_i \sin\theta_i = 8.832$$

$$\sum M_i r_i l_i \cos\theta_i = -7.589$$

$$\sum M_i r_i l_i \sin\theta_i = -1.028$$

$$\left[\left(\sum M_i r_i l_i \cos\theta_i\right)^2 + \left(\sum M_i r_i l_i \sin\theta_i\right)^2\right]^{0.5} = M_M r_M l_M$$

$$\left[(-7.589)^2 + (-1.028)^2\right]^{0.5} = 0.04M_M$$

$$M_M = \frac{7.658}{0.04} = 191.45 \text{ kg}$$

$$\tan\theta_M = \frac{-\sum M_i r_i l_i \sin\theta_i}{-\sum M_i r_i l_i \cos\theta_i}$$

$$= \frac{-(-1.028)}{-(-7.589)} = 0.13546$$

$$\theta_M = 7.71°$$

Table 10.4

Plane	M kg	r m	Mr	θ deg	$Mr \times \cos\theta$	$Mr \times \sin\theta$	l	Mrl	$Mrl \times \cos\theta$	$Mrl \times \sin\theta$
A	150	0.07	10.5	0	10.5	0	–0.1	–1.05	–1.05	0
B	200	0.06	12	45	8.485	8.485	0.2	2.40	1.697	1.697
C	300	0.05	15	105	–3.882	14.489	0.3	4.50	–1.165	4.346
D	250	0.08	20	225	–14.142	–14.142	0.5	10.0	–7.071	–7.071
L	M_L	0.10	$0.1M_L$	θ_L	$0.1M_L \times \cos\theta_L$	$0.1M_L \times \sin\theta_L$	0	0	0	0
M	M_M	0.10	$0.1M_M$	θ_M	$0.1M_M \times \cos\theta_M$	$0.1M_M \times \sin\theta_L$	0.4	$0.04M_M$	$0.04M_M \times \cos\theta_M$	$0.04M_M \times \sin\theta_M$

Since the numerator and denominator are both positive, therefore θ_M lies in the first quadrant.

$$M_L r_L = \left[\left(\sum M_i r_i \cos\theta_i + M_M r_M \cos\theta_M\right)^2 + \left(\sum M_i r_i \sin\theta_i + M_M r_M \sin\theta_M\right)^2\right]^{0.5}$$

$$0.1M_L = \left[(0.961 + 0.1 \times 191.45 \times \cos 7.71°)^2 + (8.832 + 0.1 \times 191.45 \times \sin 7.71°)2\right]^{0.5}$$

$$= \left[(19.933)^2 + (11.400)^2\right]^{0.5} = 22.963$$

$$M_L = 229.63 \text{ kg}$$

$$\tan\theta_L = \frac{-\left(\sum M_i r_i \sin\theta_i + M_M r_M \sin\theta_M\right)}{-\left(\sum M_i r_i \cos\theta_i + M_M r_M \cos\theta_M\right)}$$

$$= \frac{-11.400}{-19.933} = 0.5719; \quad \theta_L = 29.76° + 180° = 209.76°$$

Since the numerator and denominator are both negative, therefore θ_L lies in the third quadrant.

Example 10.3

A shaft has three eccentrics of mass 1 kg each. The central plane of the eccentrics is 50 mm apart. The distances of the centers from the axis of rotation are 20 mm, 30 mm and 20 mm and their angular positions are 120° apart. Find the amount of out-of-balance force and couple at 600 rpm. If the shaft is balanced by adding two masses at a radius of 70 mm and at a distance of 100 mm from the central plane of the middle eccentric, find the amount of the masses and their angular positions.

■ **Solution**

Analytical method

Let *L* and *M* be the planes at which the balancing masses are to be placed, as shown in Fig. 10.8(a). Take *L* as the reference plane.

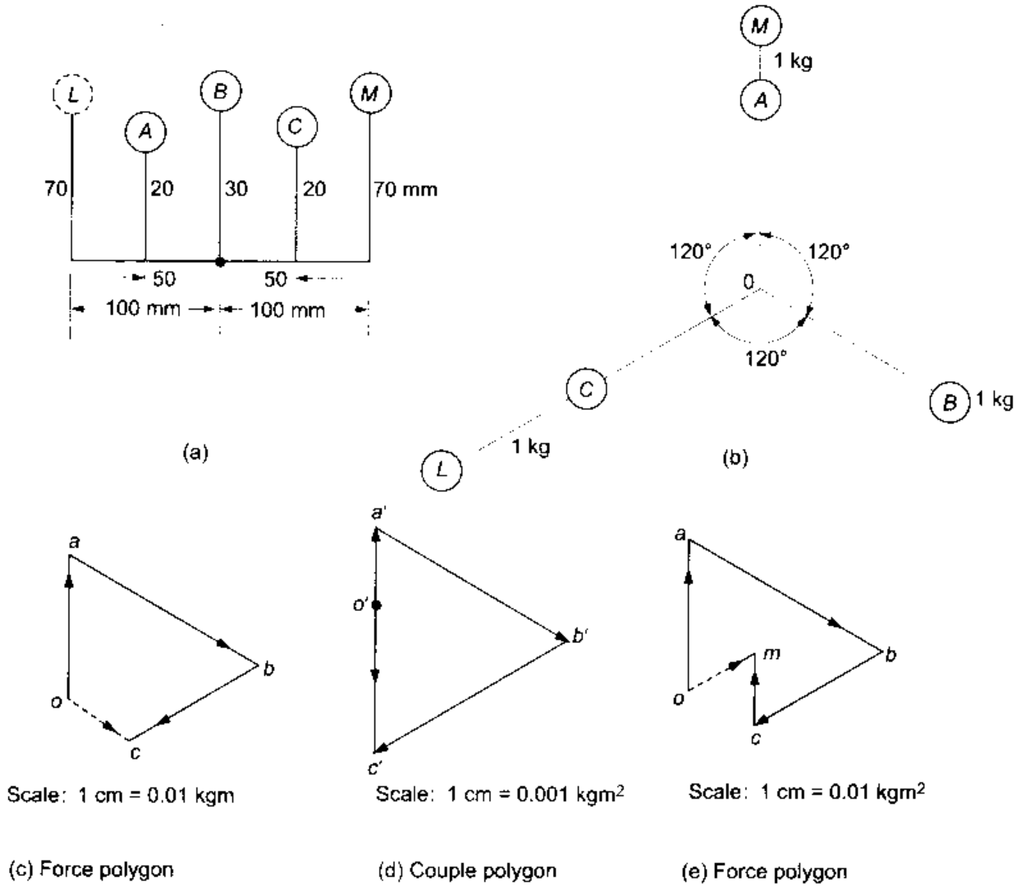


Fig.10.8

Table 10.5

Plane	M kg	r m	Mr	θ deg	Mr	$Mr \times \sin\theta$	l	Mrl	$Mrl \times \cos\theta$	$Mrl \times \sin\theta$
A	1	0.02	0.02	0	0.02	0	0.05	0.001	0.001	0
B	1	0.03	0.03	120	−0.015	0.02598	0.10	0.003	−0.0015	0.0026
C	1	0.02	0.02	240	−0.01	−0.01732	0.15	0.003	−0.0015	−0.0026
L	M_L	0.07	$0.07M_L$	θ_L	$0.07M_L \times \cos\theta_L$	$0.07M_L \times \sin\theta_L$	0	0	0	0
M	M_M	0.07	$0.07 \times M_M$	θ_M	$0.07M_M \times \cos\theta_M$	$0.07M_M \times \sin\theta_L$	0.2	$0.014 \times M_M$	$0.014M_M \times \cos\theta_M$	$0.014M_M \times \sin\theta_M$

From Table 10.5,

$$\sum M_i r_i \cos\theta_i = -0.005$$
$$\sum M_i r_i \sin\theta_i = 0.00866$$
$$\sum M_i r_i l_i \cos\theta_i = -0.005$$
$$\sum M_i r_i l_i \sin\theta_i = 0$$
$$\left[\left(\sum M_i r_i l_i \cos\theta_i\right)^2 + \left(\sum M_i r_i l_i \sin\theta_i\right)^2\right]^{0.5} = M M r_M l_M$$
$$\left[(-0.002)^2 + (0)^2\right]^{0.5} = 0.014 M_M$$
$$M_M = \frac{0.002}{0.014} = 0.1428 \text{ kg}$$
$$\tan\theta_M = \frac{-\sum M_i r_i l_i \sin\theta_i}{-\sum M_i r_i l_i \cos\theta_i} = \frac{-(0)}{-(-0.002)} = 0$$
$$\theta_M = 0° \quad \text{or} \quad 360°$$

Since the numerator is negative and denominator is positive, therefore θ_M lies in the fourth quadrant.

$$M_L r_L = \left[\left(\sum M_i r_i \cos\theta_i + M_M r_M \cos\theta_M\right)^2 + \left(\sum M_i r_i \sin\theta_i + M_M r_M \sin\theta_M\right)^2\right]^{0.5}$$
$$0.07 M_L = \left[(-0.005 + 0.07 \times 0.1428 \times \cos 0°)^2 + (0.00866 + 0.07 \times 0.1428 \times \sin 0°)2\right]^{0.5}$$
$$= \left[(0.005)^2 + (0.00866)^2\right]^{0.5} = 0.01$$
$$M_L = 0.1428 \text{ kg}$$
$$\tan\theta_L = \frac{-\left(\sum M_i r_i \sin\theta_i + M_M r_M \sin\theta_M\right)}{-\left(\sum M_i r_i \cos\theta_i + M_M r_M \cos\theta_M\right)}$$
$$= \frac{-0.00866}{-0.005 = 1.732}; \quad \theta_L = 240°$$

Since the numerator and denominator are both negative, therefore θ_L lies in the third quadrant.

Graphical method

Out-of-balance force The out-of-balance force is obtained by drawing the force polygon, as shown in Fig.10.8(c), drawn from the data in column 4 of the Table 10.5.

The resultant *oc* represents the out-of-balance force.

$$\text{Out-of-balance force} = \text{vector } oc \times \omega^2 = 0.01 \times \left(2\pi \times \frac{600}{60}\right)^2$$
$$= 39.48 \text{ N}$$

Out-of-balance couple The out-of-balance couple is obtained by drawing the couple polygon from the data in column 9, as shown in Fig.10.8(d).

$$\text{Out-of-balance couple} = o'c' \times \omega^2 = 0.002\left(2\pi \times \frac{600}{60}\right)^2$$
$$= 7.9 \text{ Nm}$$

Balancing masses The vector $c'o'$ from c' to o', as shown in Fig.10.8(d), represents the balancing couple.

$$0.014M_M = \text{vector } c'o' = 0.002$$

or

$$M_M = 0.1428 \text{ kg}$$

Draw OM parallel to $c'o'$ in Fig.10.8(b). We find that the angular position of mass M is from mass A.

To find the balancing mass M_L, draw the force polygon, as shown in Fig. 10.8(e). The closing side do of the polygon represents the balancing force.

$$0.07M_L = \text{vector } om = 0.01$$

or

$$M_L = 0.1428 \text{ kg}$$

Now draw O_L in Fig.10.8(b), parallel to om. We find that the angular position of M_L is 120° from mass A.

Example 10.4

Three masses $M_1 = 3$ kg, $M_2 = 4$ kg, and $M_3 = 3$ kg are rotating in different planes as shown in Fig.10.9. Two balancing masses are B_1 and B_2 are placed at 100 mm from each end at 80 mm radius. Find the magnitude and angular location of the balancing masses.

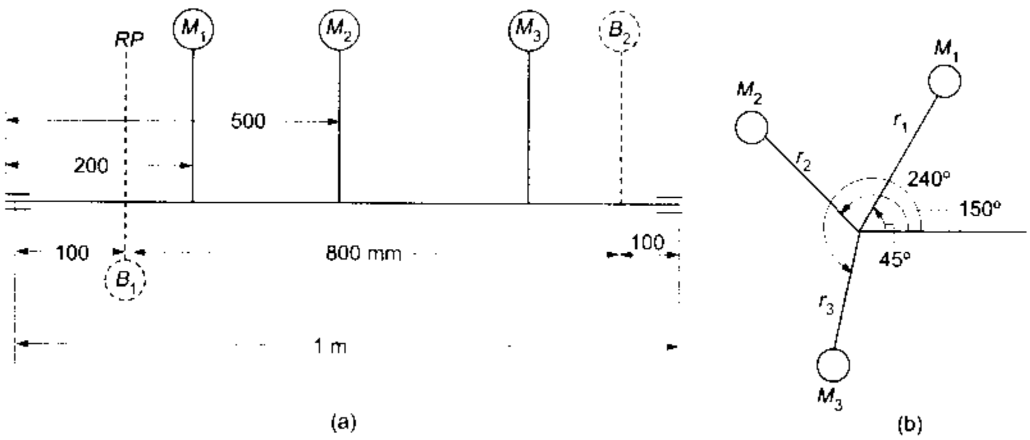


Fig.10.9 Three masses rotating in different planes

■ **Solution**

Reference plane I

From Table 10.6,

$$\sum M_i r_i \cos\theta_i = -0.2020$$

$$\sum M_i r_i \sin\theta_i = 0.2458$$

$$\sum M_i r_i l_i \cos\theta_i = -0.14973$$

$$\sum M_i r_i l_i \sin\theta_i = 0.01627$$

$$\left[\left(\sum M_i r_i l_i \cos\theta_i\right)^2 + \left(\sum M_i r_i l_i \sin\theta_i\right) 2\right]^{0.5]} = B_2 r_2 l_2$$

$$\left[(-0.14973)^2 + (0.01627)^2\right]^{0.5} = 0.064B_2$$

$$B_2 = \frac{0.15061}{0.064} = 2.35 \text{ kg}$$

$$\tan\theta_2 = \frac{-\sum M_i r_i l_i \sin\theta_i}{-\sum M_i r_i l_i \cos\theta_i}$$

$$= \frac{-(0.01627)}{-(-0.14973)} = -0.10866$$

$$\theta_2 = -6.2^\circ or 353.8^\circ \text{ ccw}$$

Table 10.6

Plane	M kg	r m	Mr	θ deg	Mr $\times\cos\theta$	Mr $\times\sin\theta$	l	Mrl	Mrl $\times\cos\theta$	Mrl $\times\sin\theta$
M_1	3	0.08	0.24	45	0.1697	0.1697	0.1	0.024	0.01697	0.01697
M_2	4	0.09	0.36	150	−0.3117	0.1800	0.4	0.144	−0.1247	0.0720
M_3	3	0.04	0.12	240	−0.0600	−0.1039	0.7	0.084	−0.0420	−0.0727
1	B_1	0.08	$0.08B_1$	θ_1	$0.08B_1$ $\times\cos\theta_1$	$0.08B_1$ $\times\sin\theta_L$	0	0	0	0
2	B_2	0.08	$0.08B_2$	θ_2	$0.08B_2$ $\times\cos\theta_2$	$0.08B_2$ $\times\sin\theta_2$	0.8	$0.064B_2$	$0.064B_2$ $\times\cos\theta_2$	$0.064B_2$ $\times\sin\theta_2$

Since the numerator is negative and denominator is positive, therefore θ_2 lies in the fourth quadrant.

$$B_1 r_1 = \left[\left(\sum M_i r_i \cos\theta_i + B_2 r_2 \cos\theta_2\right) 2 + \left(\sum M_i r_i \sin\theta_i + B_2 r_2 \sin\theta_2\right)^2\right]^{0.5}$$

$$0.08B_1 = \left[(-0.2020 + 0.08 \times 2.35 \times \cos 353.8^\circ)^2 + (0.2458 + 0.08 \times 2.35 \times \sin 353.8^\circ)^2\right]^{0.5}$$

$$= \left[(-0.0151)^2 + (0.22549)^2\right]^{0.5} = 0.226$$

$$B_1 = 2.825 \text{ kg}$$

$$\tan\theta_1 = -\left(\sum M_i r_i \sin\theta_i + B_2 r_2 \sin\theta_2\right) / -\left(\sum M_i r_i \cos\theta_i + B_2 r_2 \cos\theta_2\right)$$

$$= -0.22549/-(-0.0151) = -14.93311$$

$$\theta_1 = -86.169^\circ \quad \text{or} \quad 273.831^\circ$$

Since the numerator is negative and denominator is positive, therefore θ_1 lies in the fourth quadrant.

10.3 RECIPROCATING MASSES

10.3.1 Reciprocating Engine

Consider the reciprocating engine mechanism shown in Fig.10.10.

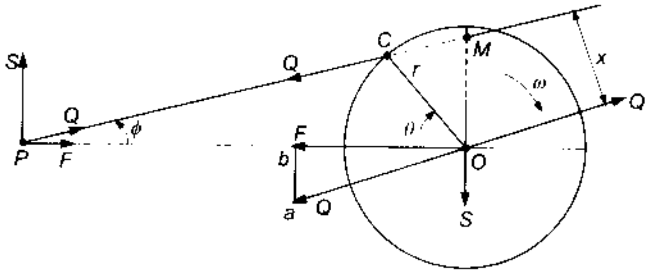


Fig.10.10 Reciprocating engine mechanism

Acceleration of the piston, $a_p = a_c\left[\cos\theta + \frac{\cos 2\theta}{n}\right]$

where a_c = acceleration of the crankpin $= \omega^2 r$,

$n = \frac{\text{Length of connecting rod}}{\text{Radius of crank}} = \frac{l}{r}$

θ = inclination of crank to inner dead center,

Q = force in the connecting rod,

S = thrust on the piston or guide bars,

x = perpendicular distance between the connecting rod and the crankpin and

Let R = mass of the reciprocating parts.

Accelerating force, $F = Ra_p$

The force Q at C is equivalent to a force Q at O and a couple Qx tending to retard the rotation of the crankshaft.

Thrust couple $= S \cdot OP$

Now triangles Oba and POM are similar. Therefore,

$$\frac{ba}{Ob} = \frac{S}{F} = \frac{OM}{OP}$$

or $$S \cdot OP = F \cdot OM$$

Also $$\frac{ba}{Oa} = \frac{S}{Q} = \frac{OM}{PM}$$

and $$\frac{x}{OM} = \cos\phi = \frac{OP}{PM}$$

so that $$\frac{OM}{PM} = \frac{x}{OP}$$

Hence $$\frac{S}{Q} = \frac{x}{OP}$$

and $$S \cdot OP = F \cdot OM = Q \cdot x$$

The full effect on the engine frame of the inertia of the reciprocating mass is equivalent to the force F along the line of stroke at O and the clockwise couple of magnitude $S \cdot OP$.

Now
$$F = Ra_c\left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$
$$= Ra_c\cos\theta + \frac{Ra_c\cos 2\theta}{n}$$
$$= F_p + F_s \qquad (10.12)$$

where F_p is the primary force, which represents the inertia force of reciprocating mass having simple harmonic motion and F_s is the secondary force, which represents the correction required to account for the obliquity of the connecting rod.

The unbalanced force due to the reciprocating mass varies in magnitude but is constant in direction. A single revolving mass cannot be used to balance a reciprocating mass, nor vice versa.

10.3.2 Partial Primary Balance

Consider the reciprocating engine mechanism shown in Fig.10.11. The primary unbalanced force,

$$F_p = Ra_c\cos\theta$$
$$= R\omega^2 r\cos\theta$$

= component parallel to the line of stroke of the centrifugal force produced by an equal mass attached to and revolving with the crankpin.

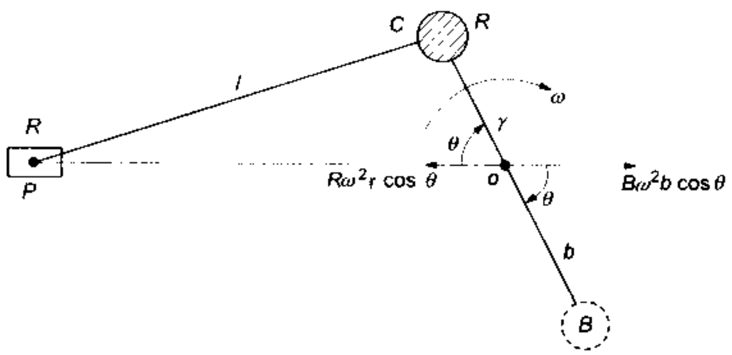


Fig.10.11 Partial balancing of primary unbalanced force

Let a balance mass B be placed along the line of crank at a radius b opposite to the crankpin. Comparing the horizontal components of forces, we have

$$R\omega^2 r\cos\theta = B\omega^2 b\cos\theta$$

or
$$Rr = Bb$$

Component of revolving mass perpendicular to the line of stroke $= B\omega^2 b\sin\theta$

This is the component of the balancing force which remains unbalanced.

It is usually preferable to make $Bb = c\cdot Rr$, where $c < 1$.

Reduced value of unbalanced force parallel to line of stroke

$$= (1-c)R\omega^2 r\cos\theta \qquad (10.13)$$

Unbalanced force perpendicular to line of stroke $= cR\omega^2 r\sin\theta$ (10.14)

For unbalanced force to be least, $c = 0.5$

$$\text{Resultant unbalanced force} = R\omega^2 r\left[(1-c)^2\cos^2\theta + c^2\sin^2\theta\right]^{0.5} \quad (10.15)$$

If the balance mass B has to balance the revolving parts M as well as give a partial balance of the reciprocating parts R, then

$$Bb = Mr + cRr = (M + cR)r \quad (10.16)$$

In practice, two balance masses, each equal to $\frac{B}{2}$, would be attached to the crank webs. The graphical representation of the various forces is shown in Fig.10.12,

where oa = primary disturbing force

ob = centrifugal force due to the revolving balance mass

oc = residual unbalanced force parallel to the line of stroke

oe = unbalanced force at right angles to the line of stroke

of = resultant unbalanced force on engine frame

Let $od = oa/2$, for 50% balancing of reciprocating parts

Then $oc = od$ and $\angle cof = \theta$.

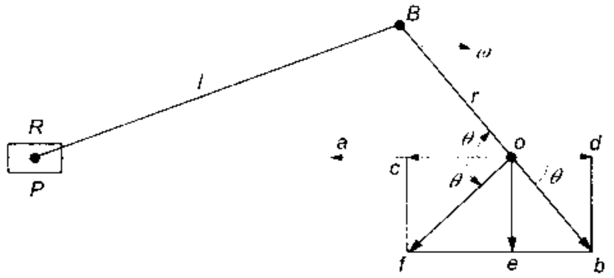


Fig.10.12 Graphical representation of forces

Example 10.5

A single cylinder reciprocating engine has speed 240 rpm, stroke 300 mm, mass of the reciprocating parts 50 kg and mass of the revolving parts 40 kg at 150 mm radius. If two-third of the reciprocating parts and all the revolving parts are to be balanced, find (a) the balance mass required at a radius of 400 mm, and (b) the residual unbalanced force when the crank has rotated 60° from top dead centre.

■ **Solution**

(a)
$$Bb = (M + cR)r$$

$$0.4B = \left(40 + \frac{2 \times 50}{3}\right)0.15$$

or
$$B = 27.5 \text{ kg}$$

(b)
$$\text{Residual unbalanced force} = R\omega^2 r\left[(1-c)^2\cos^2\theta + c^2\sin^2\theta\right]^{0.5}$$

$$= 50 \times \left(2\pi \times \frac{240}{60}\right)^2 \times 0.15\left[\left(1 - \frac{2}{3}\right)^2\cos^2 60° + \left(\frac{2}{3}\right)^2 \times \sin^2 60°\right]^{0.5}$$

$$= 2846.9 \text{ N}$$

10.4 PARTIAL BALANCING OF UNCOUPLED LOCOMOTIVES

In an uncoupled locomotive, two cylinders are placed symmetrically either inside or outside the frames. The two cranks are at right angles to each other, as shown in Fig.10.13(a). In an uncoupled locomotive, the effort is transmitted to one pair of wheels only, whereas in a coupled locomotive, the driving wheels are connected to the leading and trailing wheel by an outside coupling rod. $c \approx \frac{2}{3}$ to $\frac{3}{4}$ with two pairs of coupled wheels and $c = \frac{2}{5}$ for four cylinder locomotives.

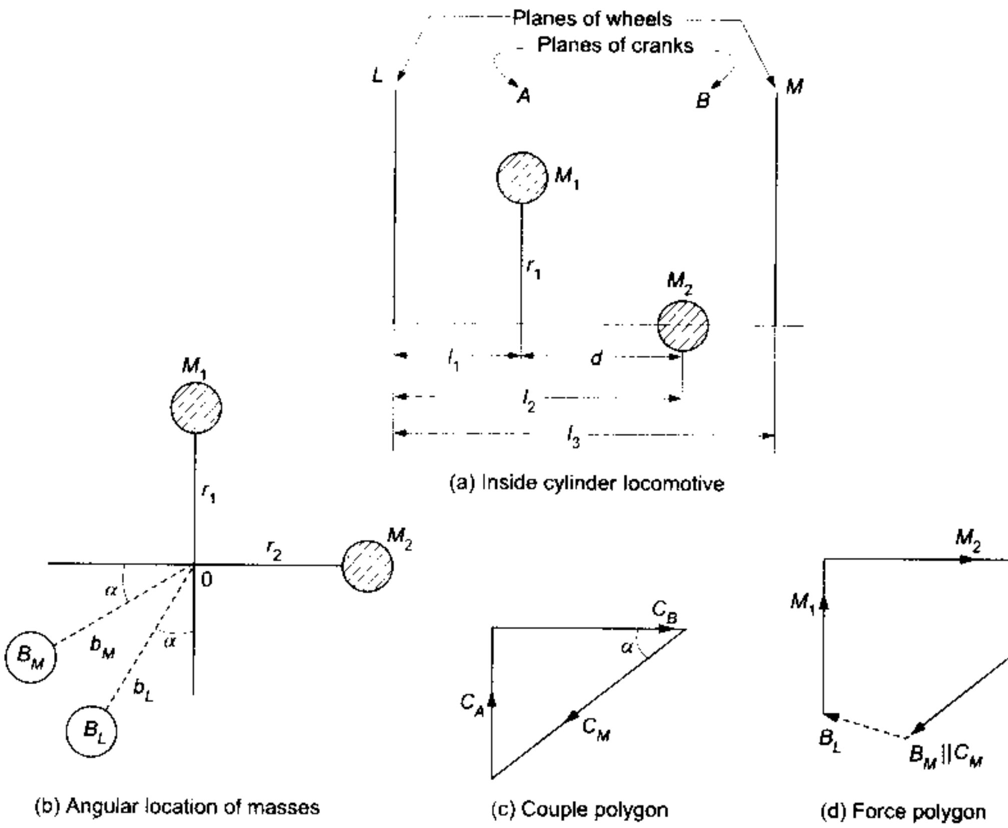


Fig.10.13 Uncoupled locomotive

The location of the cranks and balance masses is shown in Fig.10.13(b), couple polygon in Fig.10.13(c) and force polygon in Fig.10.13(d). The couple polygon may be drawn by using the Table 10.7.

Table 10.7

Plane	Mass M	Radius r	$M \cdot r$	Distance from plane L l	Couple Mrl
L	B_L	b_L	$B_L b_L$	0	0
A	M_1	r_1	$M_1 r_1$	l_1	$C_A = M_1 r_1 l_1$
B	M_2	r_2	$M_2 r_2$	l_2	$C_B = M_2 r_2 l_2$
M	B_M	b_M	$B_M b_M$	l_3	$C_M = B_M b_M l_3$

From Table 10.7,

$$C_M = \left[C_A^2 + C_B^2\right]^{0.5} = B_M b_M l_3$$

or $$B_M = \frac{C_M}{b_M l_3}; \qquad \tan\alpha = \frac{C_A}{C_B}$$

If the mass of revolving parts to be balanced = M
and the mass of reciprocating parts to be balanced = R
Then total equivalent mass of revolving parts to be balanced,

$$M_1, M_2 = M + cR$$

Part of each balance mass required for reciprocating masses,

$$B_r = cR\frac{B_M}{M_1}$$

Then draw force polygon to determine B_L.

Hammer blow The unbalanced force perpendicular to the line of stroke due to balance mass B_r at radius b to balance the reciprocating parts only is equal to $B_r\omega^2 b\sin\theta$. The maximum magnitude of this force is known as hammer blow. This occurs at $\theta = 90°$ and 270°.

$$\text{Hammer blow} = B_r\omega^2 b \tag{10.17}$$

If P is the downward pressure on rails due to dead load. Then

$$\text{Net pressure} = P \pm B_r\omega^2 b$$

$$\text{Permissible speed, } \omega = \sqrt{\frac{P}{B_r b}} \tag{10.18}$$

Variation of tractive effort

$$\text{The variation of tractive effort} = (1-c)R\omega^2 r[\cos\theta + \cos(90° + \theta)]$$
$$= (1-c)R\omega^2 r[\cos\theta - \sin\theta]$$

For its value to be maximum, $(d/d\theta)(\cos\theta - \sin\theta) = 0$

or $$-\sin\theta - \cos\theta = 0$$

or $$\tan\theta = -1$$

or $$\theta = -45° \quad \text{and} \quad +135°$$

$$\text{Maximum variation of tractive effort} = \pm\sqrt{2}(1-c)R\omega^2 r \tag{10.19}$$

Swaying couple The unbalanced part of the primary disturbing forces cause a horizontal swaying couple to act on the locomotive owing to the distance d between the cylinder centres.

Taking moments about the engine centre line, the resultant unbalanced couple is:

$$(1-c)R\omega^2 r\cdot\frac{d}{2}\cdot[\cos\theta - \cos(90° + \theta)]$$

$$\text{Swaying couple} = (1-c)R\omega^2 r\cdot\frac{d}{2}\cdot[\cos\theta + \sin\theta]$$

This is maximum when $\theta = 45°$ and 225°

$$\text{Maximum swaying couple} = \pm\left[\frac{1-c}{\sqrt{2}}\right]R\omega^2 r\cdot d$$

Example 10.6

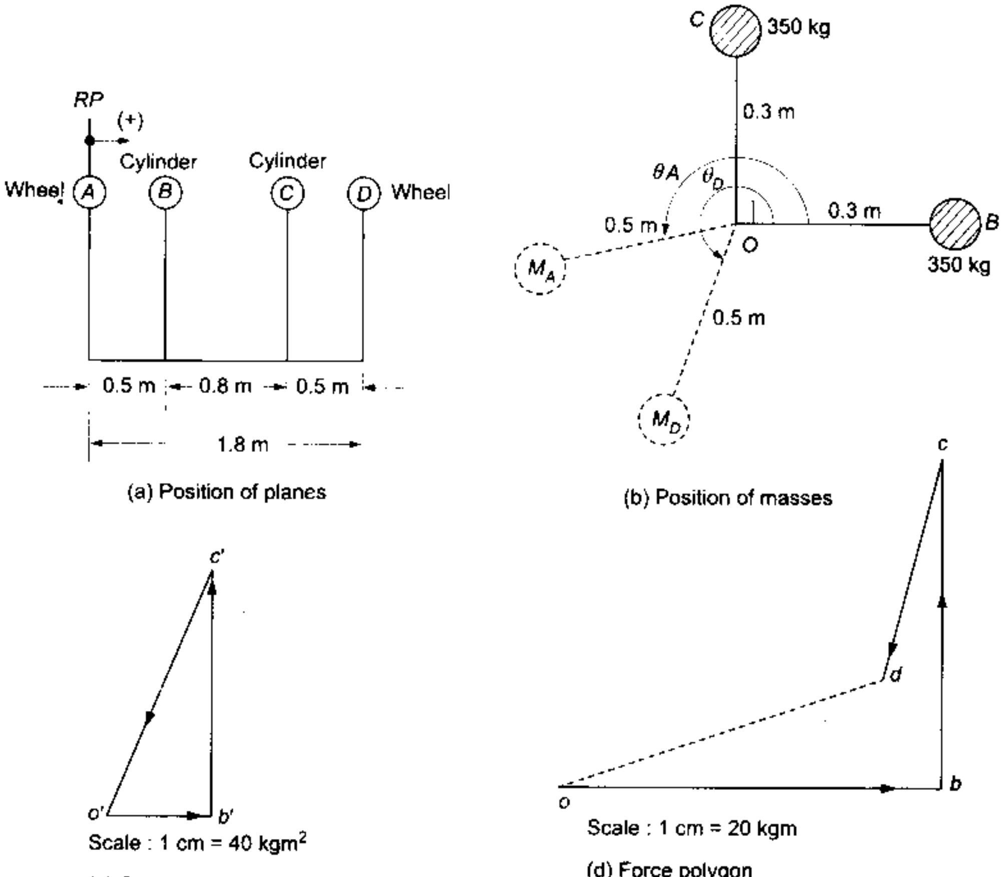
An inside cylinder locomotive has its cylinder centre lines 0.8 m apart and has a stroke of 0.6 m. The rotating masses are equivalent to 150 kg at the crank pin and the reciprocating masses per cylinder are 300 kg. The wheel centre lines are 1.8 m apart. The cranks are at right angles. The whole of the rotating and 2/3rd of the reciprocating masses are to be balanced by masses placed at a radius of 0.5 m. Find (a) the magnitude and direction of the balancing masses, (b) the fluctuation in rail pressure under one wheel, (c) the variation of tractive effort, and (d) the magnitude of swaying couple at a crank speed of 300 rpm.

■ **Solution**

$$\text{Equivalent mass to be balanced} = 150 + 2 \times \frac{300}{3} = 350 \text{ kg}$$

Balancing masses

Let M_A and M_D be the balancing masses at angular locations θ_A and θ_D respectively. The position of the planes is shown in Fig.10.14(a) and angular position of the masses in Fig. 10.14(b). Take *A* as the reference plane.



(a) Position of planes

(b) Position of masses

(c) Couple polygon

(d) Force polygon

Fig.10.14

Table 10.8

Plane	Mass M	Radius r	$M \cdot r$	Distance from plane A l	Couple Mrl
A	M_A	0.5	$0.5M_A$	0	0
B	350	0.3	105	0.5	52.5
C	350	0.3	105	1.3	136.5
D	M_D	0.5	$0.5M_D$	1.8	$0.9M_D$

Now draw the couple polygon, as shown in Fig.10.14(c), from the data in column 6 of Table 10.8. The closing side *co* represents the balancing couple.

$$0.9M_D = \text{vector } c'o' = 3.7 \times 40 = 148$$
$$M_D = 164.4 \text{ kg}$$

Draw *oD* parallel to *c'o'* in Fig.10.14(b). By measurement, $\theta_D = 250°$.

To find the balancing mass M_A, draw the force polygon, as shown in Fig.10.14(d), from the data in column 4. The vector *do* represents the balancing force.

$$0.5M_A = \text{vector } od = 4.1 \times 20 = 82$$

or
$$M_A = 164 \text{ kg}$$

To find the angle θ_A, draw *oA* parallel to *od* in Fig.10.14(b). By measurement, $\theta_A = 200°$.

Analytically
$$0.9M_D = \left[(52.5)^2 + (136.5)^2\right]^{0.5} = 146.248$$
$$M_D = 162.5 \text{ kg}$$
$$\theta_D = 180° + \tan^{-1}\left(\frac{136.5}{52.5}\right)$$
$$= 180° + 68.96° = 248.96° \text{ ccw}$$

Similarly,
$$0.9M_A = \left[(136.5)^2 + (52.5)^2\right]^{0.5} = 146.248$$
$$M_A = 162.5 \text{ kg}$$
$$\theta_D = 180° + \tan^{-1}\left(\frac{52.5}{136.5}\right)$$
$$= 180° + 21.03° = 201.03° \text{ counter-clockwise}$$

(c) Each balance mass = 162.5 kg

Balance mass for the rotating masses $= 150 \times \dfrac{162.5}{300} = 81.25$ kg

Balance mass for the reciprocating masses, $B = \left(\dfrac{2}{3}\right) \cdot \left(\dfrac{300}{350}\right) \times 162.5 = 92.86$ kg

Fluctuation in the rail pressure or hammer blow $= B\omega^2 b = 92.86 \times \left(2\pi \times \dfrac{300}{60}\right)^2 \times 0.5$

$$= 45824 \text{ N}$$

(d) Maximum variation of tractive effort $= \pm\sqrt{2}(1-c)R\omega^2 r$

$$= \pm\sqrt{2}\left(1-\frac{2}{3}\right)300 \times \left(2\pi \times \frac{300}{60}\right)^2 0.3$$

$$= \pm 41873 \text{ N}$$

(e) Maximum swaying couple $= \pm\left[\dfrac{1-c}{\sqrt{2}}\right]R\omega^2 r \cdot d$

$$= \pm\left[\frac{1-\frac{2}{3}}{\sqrt{2}}\right] \times 300(10\pi)^2 \times 0.3 \times 0.8$$

$$= \pm 16749 \text{ Nm}$$

Example 10.7

The following data refer to an outside cylinder uncoupled locomotive:

Mass of rotating parts per cylinder = 350 kg

Mass of reciprocating parts per cylinder = 300 kg

Angle between cranks = 90°

Crank radius = 0.3 m

Cylinder centers = 1.8 m

Radius of balance masses = 0.8 m

Wheel centers = 1.5 m

If whole of the rotating and 2/3rd of the reciprocating parts are to be balanced in planes of the driving wheels, find (a) magnitude and angular positions of balance masses, (b) speed in km/h at which the wheel will lift off the rails when the load on each driving wheel is 30 kN, and the diameter of tread of driving wheels is 1.8 m, and (c) swaying couple at speed found in (b) above.

■ **Solution**

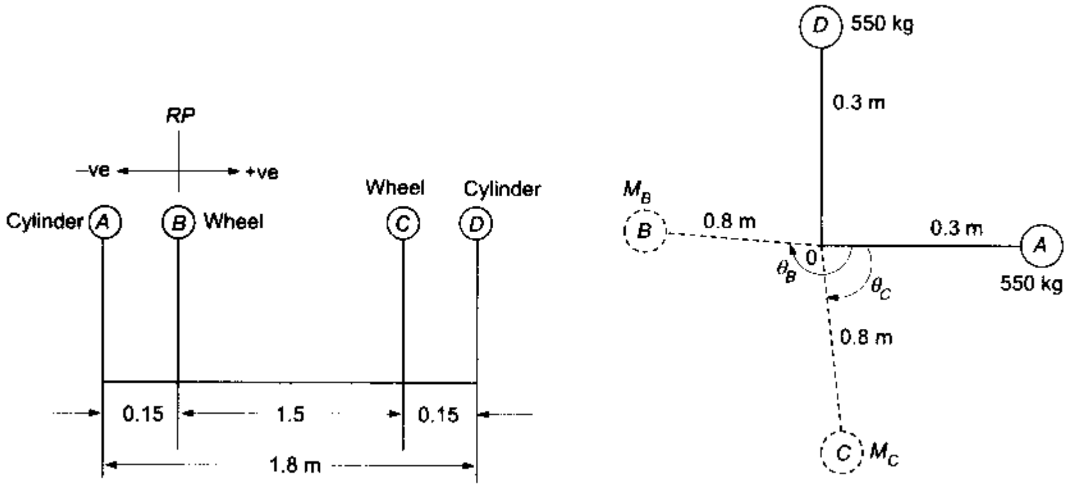
(a) Equivalent mass of the rotating parts to be balanced per cylinder,

$$M = 350 + 2 \times \frac{300}{3} = 550 \text{ kg}$$

Let M_B and M_C be the balance masses, and θ_B and θ_C their angular positions. Let B be the reference plane. The position of planes is shown in Fig.10.15(a), and the position of masses in Fig.10.15(b).

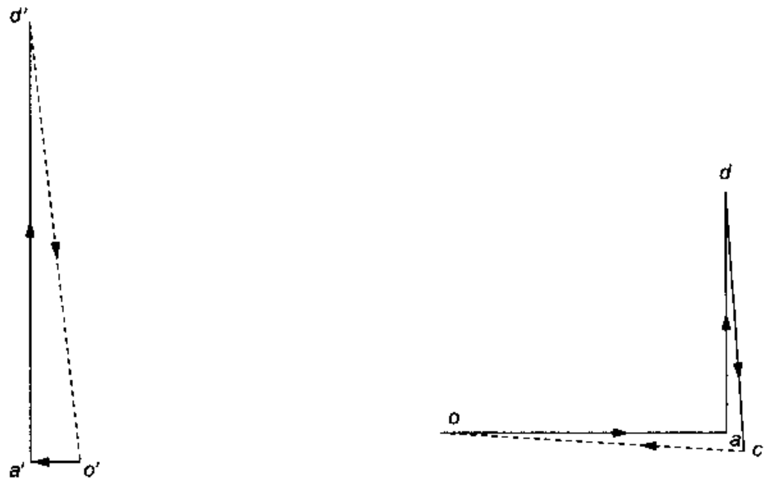
Table 10.9

Plane	Mass M	Radius r	$M \cdot r$	Distance from plane B l	Couple Mrl
A	550	0.3	165	– 0.15	–24.75
B	M_B	0.8	$0.8M_B$	0	0
C	M_C	0.8	$0.8M_C$	1.5	$1.2M_C$
D	550	0.3	165	1.65	272.25



(a) Position of planes

(b) Mass positions



Scale: 1 cm = 50 kgm²

(c) Couple polygon

Scale: 1 cm = 50 kgm

(d) Force polygon

Fig.10.15

Draw the couple polygon, as shown in Fig.10.15(c), from the data in column 6 of Table 10.9. The closing side *do* represents the balancing couple.

$$1.2M_C = \text{vector} d'o' = 5.5 \times 50$$

or $$M_C = 229.17 \text{ kg}$$

Now draw *OC* parallel to $d'o'$ in Fig.10.15(b) to find θ_C. By measurement, $\theta_C = 84°$.

To find the balancing mass M_B, draw the force polygon, as shown in Fig.10.15(d), from the data in column 4 of the above table.

$$0.8M_B = \text{vector } co = 3.7 \times 50$$

or $$M_B = 231.25 \text{ kg}$$

Now draw *OB* parallel to co in Fig.10.15(b) to find θ_B. By measurement, $\theta_B = 186°$.

Analytically,

$$1.2M_C = \left[(24.75)^2 + (272.25)^2\right]^{0.5} = 273.37$$

$$M_C = 227.8 \text{ kg}$$

$$\theta_c = \tan^{-1}\left(\frac{272.25}{24.75}\right) = \tan^{-1} 11 = 84.8° \text{ cw}$$

$$M_B = M_C$$

$$\theta_B = 180° + \tan^{-1}\left(\frac{24.75}{272.25}\right)$$

$$= 180° + 5.2° = 185.2°$$

(b) $M_B = M_C = 227.8$ kg

Balancing mass for reciprocating parts,

$$B = \left(\frac{cR}{M}\right) M_B = \left[\frac{2 \times 300}{3 \times 550}\right] \times 227.8 = 82.83 \text{ kg}$$

$$\omega = \left[\frac{P}{B \cdot b}\right]^{0.5} = \left[\frac{30 \times 10^3}{82.83 \times 0.8}\right]^{0.5}$$

$$= 21.28 \text{ rad/s}$$

$$v = \omega \times \frac{D}{2} = 21.28 \times 0.9 \times \frac{3600}{1000} = 68.94 \text{ km/h}$$

(c) Maximum swaying couple

$$= \pm\left[\frac{1-c}{\sqrt{2}}\right] R\omega^2 r \cdot d$$

$$= \pm\left[\frac{1-2/3}{\sqrt{2}}\right] 300(21.28)^2 0.3 \times 1.8 = 17.293 \text{ kNm}$$

Example 10.8

The following data rafer to a two cylinder uncoupled locomotive:

Rotating mass per cylinder = 300 kg

Reciprocating mass per cylinder = 330 kg

Distance between the wheels = 1.4 m

Distance between the cylinder centres = 0.6 m

Diameter of treads of the driving wheels = 1.8 m

Crank radius = 0.3 m

Radius of centre of the balance mass = 0.6 m

Speed of the locomotive = 45 km/h

Angle between the cylinder cranks = 90°

Dead load of each wheel = 40 kN

Determine

(a) the balancing mass required in the planes of driving wheels if the complete revolving and 2/3rd of the reciprocating masses are to be balanced

(b) swaying couple

(c) variation in tractive effort

(d) maximum and minimum pressure on rails and

(e) maximum speed of locomotive without lifting the wheels from the rails.

■ **Solution**
(a) Mass to be balanced = 300 = 2 × 330/3 = 520 kg

Taking 1 as the reference plane in Fig.10.16

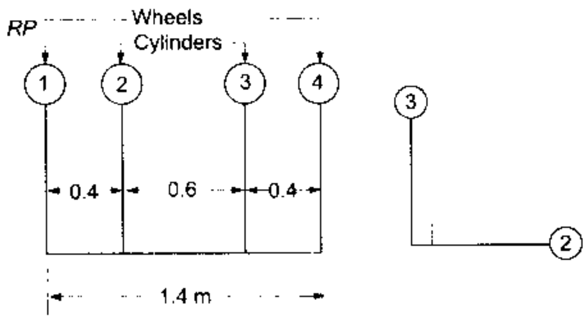


Fig.10.16 Two cylinder uncoupled locomotive

Table 10.10

Plane	M kg	r m	Mr	θ deg	Mr $\cos\theta$	Mr $\sin\theta$	l	Mrl	Mrl $\times \cos\theta$	Mrl $\times \sin\theta$
M_2	520	0.3	156	0	156	0	0.4	62.4	62.4	0
M_3	520	0.3	156	90	0	156	1.0	156	0	156
M_1	M_1	0.6	$0.6B_1$	θ_1	$0.6M_1$ $\times \cos\theta_1$	$0.6M_1$ $\times \sin\theta_1$	0	0	0	0
M_4	M_4	0.6	$0.6B_4$	θ_4	$0.6M_2$ $\times \cos\theta_4$	$0.08M_4$ $\times \sin\theta_4$	1.4	$0.84M_4$	$\theta_4 s$ $\times \cos\theta_4$	θ_4 sin $\times \sin\theta_4$

From Table 10.10,

$$\sum M_i r_i \cos\theta_i = 156$$

$$\sum M_i r_i \sin\theta_i = 156$$

$$\sum M_i r_i l_i \cos\theta_i = 62.4$$

$$\sum M_i r_i l_i \sin\theta_i = 156$$

$$\left[\left(\sum M_i r_i l_i \cos\theta_i\right)^2 + \left(\sum M_i r_i l_i \sin\theta_i\right)^2\right]^{0.5} = M_4 r_4 l_4$$

$$\left[(62.4)^2 + (156)^2\right]^{0.5} = 0.84M_4$$

$$M_4 = \frac{168.02}{0.84} = 200 \text{ kg}$$

$$\tan\theta_4 = \frac{-\sum M_i r_i l_i \sin\theta_i}{-\sum M_i r_i l_i \cos\theta_i} = \frac{-(156)}{-(62.4)} = 2.5$$

$$\theta_4 = 180 + 68.2° = 248.2° \text{ ccw}$$

Since the numerator and denominator are both negative, therefore θ_4 lies in the third quadrant.

$$M_1 r_1 = \left[\left(\sum M_i r_i \cos\theta_i + M_4 r_4 \cos\theta_4\right)^2 + \left(\sum M_i r_i \sin\theta_i + M_4 r_4 \sin\theta_4\right)^2\right]^{0.5}$$

$$0.6M_1 = \left[(156 + 0.6 \times 200 \times \cos 248.2°)^2 + (156 + 0.6 \times 200 \times \sin 248.2°)^2\right]^{0.5}$$

$$= \left[(111.43)^2 + (44.58)^2\right]^{0.5} = 120.02$$

$$M_1 = 200 \text{ kg}$$

$$\tan\theta_1 = \frac{-\left(\sum M_i r_i \sin\theta_i + M_4 r_4 \sin\theta_4\right)}{-\left(\sum M_i r_i \cos\theta_i + M_4 r_4 \cos\theta_4\right)} = \frac{-44.58}{-111.43} = 0.4$$

$$\theta_1 = 180° + 21.8° = 201.8°$$

Since the numerator and denominator are both negative, therefore θ_1 lies in the third quadrant.

(b)

$$v = 45 \times \frac{1000}{3600} = 12.5 \text{ m/s}$$

$$\omega = \frac{v}{r} = \frac{12.5}{0.9} = 13.889 \text{ rad/s}$$

Swaying couple

$$= \left[\frac{1-c}{\sqrt{2}}\right] \times Rr\omega^2 l$$

$$= \left[\frac{1-2/3}{\sqrt{2}}\right] \times 300 \times 0.3 \times (13.889)^2 \times 0.6$$

$$= 2455.3 \text{ Nm}$$

(c) Variation in tractive effort

$$= \pm\sqrt{2}(1-c)R\omega^2 r$$

$$= \pm\sqrt{2}\left(1 - \frac{2}{3}\right) \times 300 \times (13.889)^2 \times 0.3$$

$$= \pm 8184.2 \text{ N}$$

(d) Balance mass for the reciprocating parts only, $R_1 = 200 \times \frac{2}{3} \times \frac{330}{520} = 84.6$ kg

Hammer blow $= R_1 b\omega^2 = 84.6 \times 0.6 \times (13.889)^2 = 9791.8$ N

Dead weight $= 40$ kN

Maximum pressure on rails $= 40,000 + 9791.8 = 49791.8$ N

Minimum pressure on rails $= 40,000 - 9791.8 = 30208.2$ N

(e) Let ω_1 be the speed, then $84.6 \times 0.6 \times \omega_1^2 = 40,000$

$$\omega_1 = 28.07 \text{ rad/s}$$

$$v = 28.07 \times 0.9 \times \frac{3600}{1000} = 90.95 \text{ km/h}$$

10.5 COUPLED LOCOMOTIVES

In a coupled locomotive, the driving wheels are connected to the leading and trailing wheels by an outside coupling rod, as shown in Fig. 10.17. By such an arrangement, a greater portion of the engine mass is utilised for tractive purposes. The coupling rod cranks are placed diametrically opposite to the adjacent driving cranks.

The coupling rods together with cranks and pins may be treated as rotating masses and completely balanced by masses in the respective wheels. Therefore, in a coupled locomotive, the rotating and reciprocating masses must be treated separately and the balanced masses for the two systems are then suitably combined in the wheel. The hammer blow may also be considerably reduced.

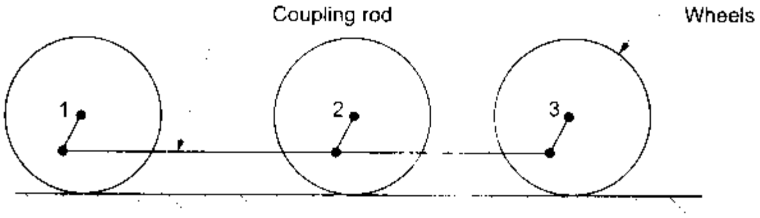


Fig.10.17 Coupled locom...

Figure 10.18(a) shows the arrangement of coupling rods, wheels and cylinders of a coupled locomotive, Fig.10.18(b) the angular position of cranks and coupling pin, Fig.10.18(c) the couple polygon when wheel E is driving and Fig.10.18(d) the force polygon when wheel B is the driver.

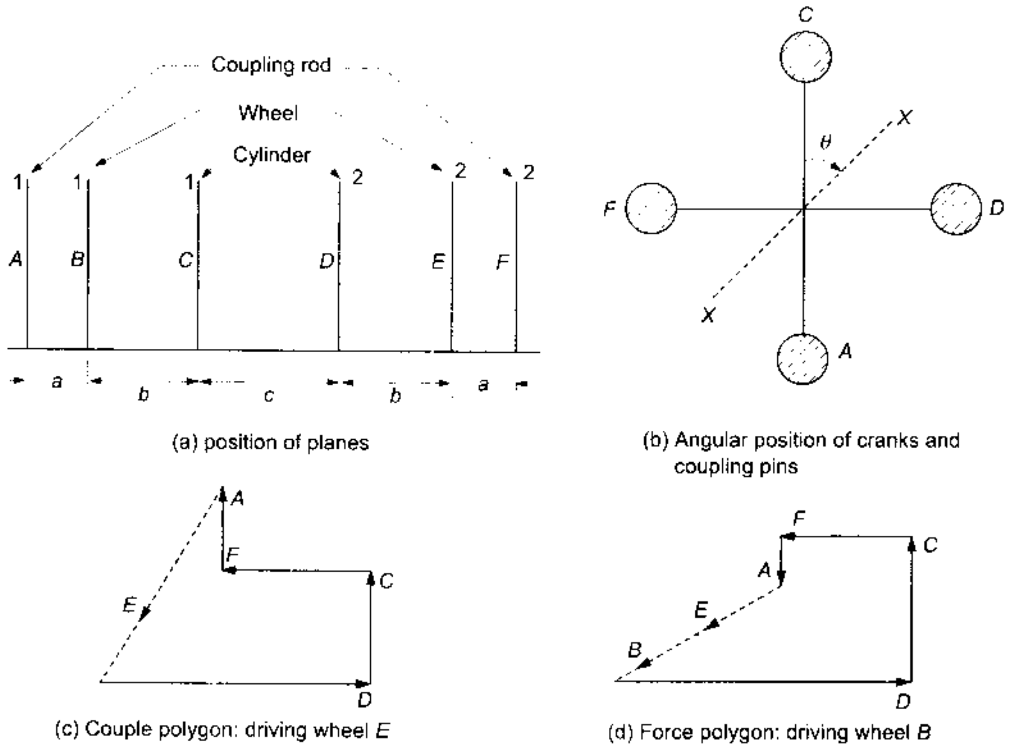


Fig.10.18 Coupled locomotive

Example 10.9

The following data refer to a two cylinder locomotive with two coupled wheels on each side:

Length of stroke = 600 mm; Mass of reciprocating parts = 280 kg

Mass of revolving parts = 200 kg; Mass of each coupling rod = 240 kg

Radius of centre of coupling rod pin = 250 mm

Distance between cylinders = 0.6 m

Distance between wheels = 1.5 m

Distance between coupling rods = 1.8 m

The main cranks are at right angles and the coupling rod pins are at 180° to their respective main cranks. The balance masses are to be placed in the wheels at a mean radius of 670 mm in order to balance the complete revolving and 3/4th of the reciprocating masses. The balance mass for the reciprocating masses is divided equally between the driving wheels and the coupled wheels. Find (a) the magnitude and angular positions of the masses required for the driving and trailing wheels, and (b) the hammer blow at 120 km/h, if the wheels are 1.8 m diameter.

■ Solution

(a) The position of planes for the driving wheels B and E, cylinders C and D, and coupling rods A and F, are shown in Fig.10.19(a). The angular position of cranks C and D and coupling pins A and F are shown in Fig.10.19(b).

Mass of the reciprocating parts per cylinder to be balanced $= 3 \times \frac{280}{4} = 210$ kg

Mass to be balanced for driving wheels and trailing wheels $= \frac{210}{2} = 105$ kg

Masses to be balanced for each driving wheel:

1. Half of the mass of coupling rod $= \frac{240}{2} = 120$ kg or $M_A = M_F = 120$ kg
2. Complete the revolving mass (200 kg) and 3/4th the mass of reciprocating parts (105 kg).

or $$M_C = M_D = 200 + 105 = 305 \text{ kg}$$

Driving wheels Let M_B and M_E be the balance masses placed in the driving wheels B and E, respectively in plane of B as the reference plane.

Table 10.11

Plane	Mass M, kg	Radius r, m	$M \cdot r$ kgm	Distance from plane B l, m	Couple Mrl, kgm^2
A	120	0.25	30.0	– 0.15	–4.5
B	M_B	0.67	$0.67M_B$	0	0
C	305	0.30	91.5	0.45	41.175
D	305	0.30	91.5	1.05	96.075
E	M_E	0.67	$0.67M_E$	1.5	$1.005M_E$
F	120	0.25	30.0	1.65	49.5

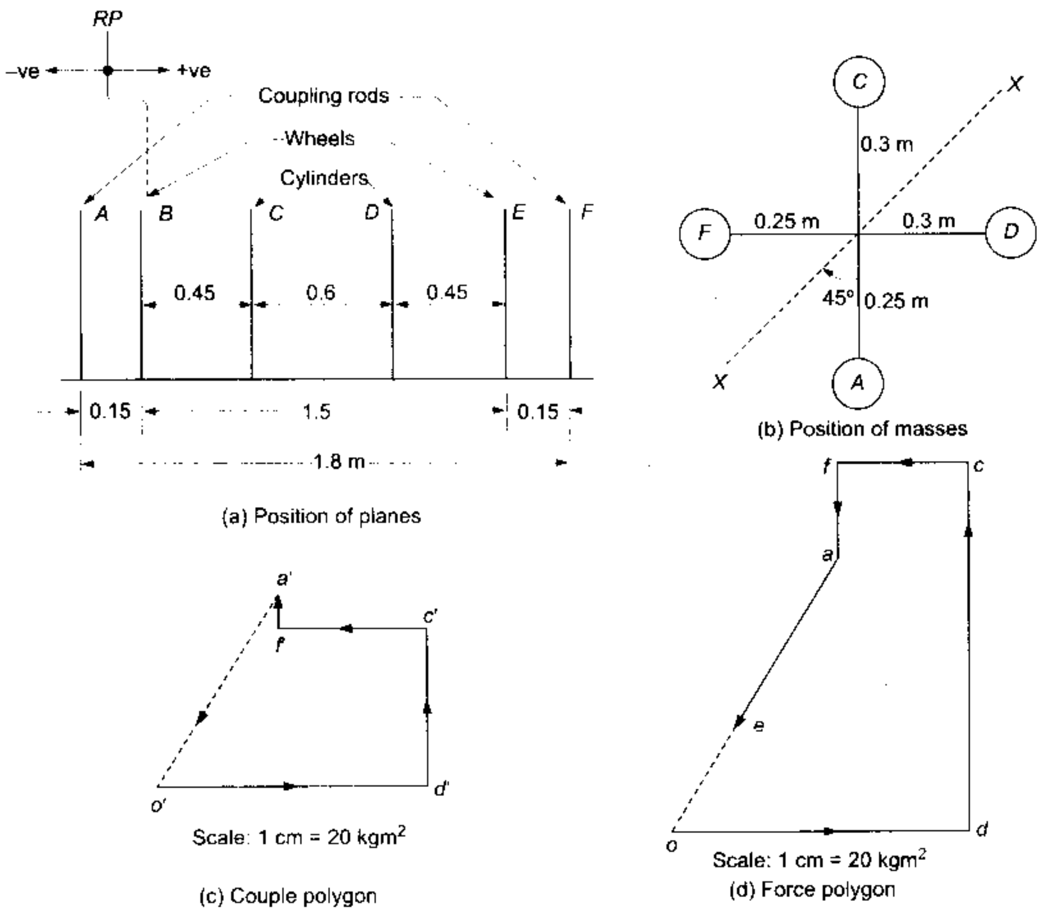


Fig.10.19

Draw the couple polygon from the data in column 6 of Table 10.11, as shown in Fig.10.19(c).

$$1.005 M_E = \text{vector } a'o' = 3.3 \times 20$$

or $$M_E = 65.67 \text{ kg}$$

and $$\theta_E = 45^\circ$$

Now draw the force polygon from the data in column 4 of the Table 10.11, as shown in Fig.10.19(d).

$$0.67 M_B = \text{vector } eo = 2.2 \times 20$$

or $$M_B = 65.67 \text{ kg}$$

and $$\theta_B = 45^\circ$$

Trailing wheels The following masses are to be balanced for each trailing wheel:

1. Half of the mass of the coupling rod. $M_A = M_F = 120$ kg
2. Mass of the reciprocating parts. $M_C = M_D = 105$ kg

Let M_B and M_E be the balanced masses placed in the trailing wheels. We take wheel B as the reference plane.

Table 10.12

Plane	Mass M, kg	Radius r, m	$M \cdot r$ kgm	Distance from plane B l, m	Couple Mrl, kgm^2
A	120	0.25	30.0	−0.15	−4.5
B	M_B	0.67	$0.67M_B$	0	0
C	105	0.30	31.5	0.45	14.175
D	105	0.30	31.5	1.05	33.075
E	M_E	0.67	$0.67M_E$	1.5	$1.005M_E$
F	120	0.25	30.0	1.65	49.5

Draw the couple polygon from the data in column 6 from Table 10.12, as shown in Fig.10.20(a).

$$1.005M_E = \text{vector } a'o' = 2.55 \times 10$$

or $$M_E = 25.37 \text{ kg}$$

and $$\theta_E = 41°$$

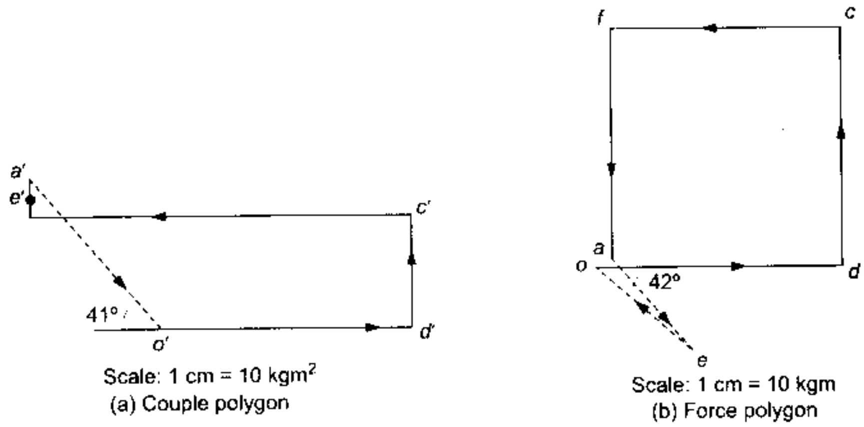


Fig.10.20

Now draw the force polygon from the data in column 4 from the above table, as shown in Fig.10.20(b).

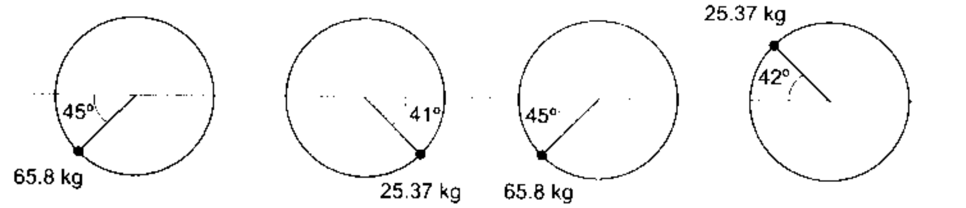
$$0.67M_B = \text{vector } eo = 1.7 \times 10$$

or $$M_B = 25.37 \text{ kg}$$

and $$\theta_B = 42°$$

The balance masses in all the four wheels are shown in Fig.10.21.

(b) To find the hammer blow, we find the balance mass required for the reciprocating masses only.



(a) Driving wheel *E* (b) Training wheel *E* (c) Driving wheel *B* (d) Training wheel *B*

Fig.10.21 Location and magnitude of balance masses

Table 10.13

Plane	Mass M, kg	Radius r, m	$M \cdot r$ kgm	Distance from plane B l, m	Couple Mrl, kgm^2
B	M_B	0.67	$0.67M_B$	0	0
C	105	0.30	31.5	0.45	14.175
D	105	0.30	31.5	1.05	33.075
E	M_E	0.67	$0.67M_E$	1.5	$1.005M_E$

Draw the couple polygon from the data in column 6 of Table 10.13, as shown in Fig.10.22.

$$1.005M_E = \text{vector } c'o = 3.6 \times 10$$

or $M_E = 35.8$ kg

Linear speed of the wheel $= 120$ km/h $= 33.33$ m/s

Diameter of wheel, $D = 1.8$ m

Angular speed of wheel, $\omega = \dfrac{2v}{D} = 2 \times \dfrac{33.33}{1.8} = 37$ rad/s

Hammer blow $= \pm B\omega^2 b = 35.8 \times (37)^2 \times 0.67 = 32836.8$ N

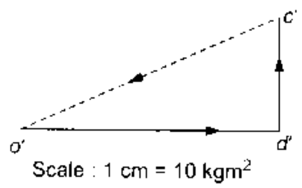


Fig.10.22 Couple polygon

Example 10.10

The following data refer to a four-coupled wheel locomotive with two inside cylinders, as shown in Fig.10.23:

Reciprocating mass per cylinder = 300 kg

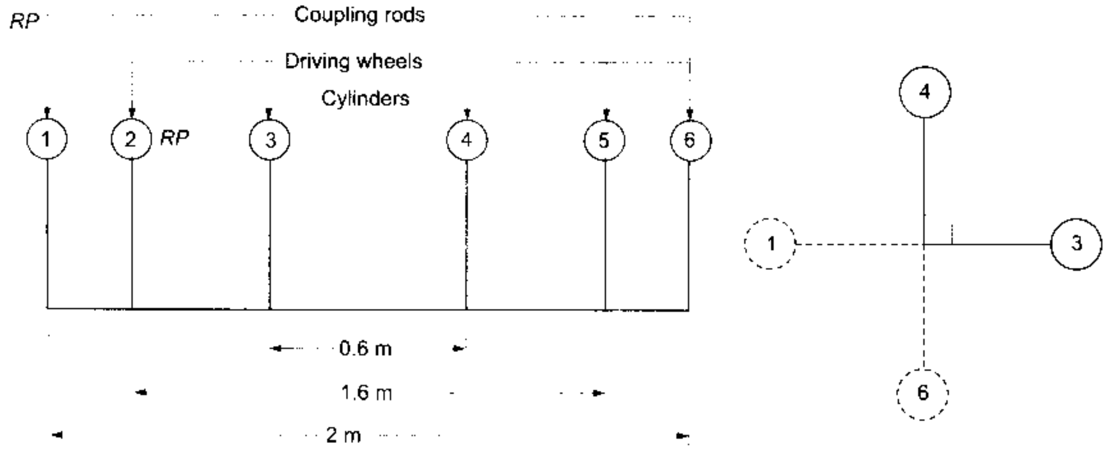
Revolving mass per cylinder = 250 kg

Diameter of the driving wheel = 1.9 m

Revolving parts for each coupling rod crank = 120 kg

Engine crank radius = 0.3 m

Coupling rod crank radius = 0.25 m



(a) (b)

Fig.10.23 Four-coupled wheel locomotive

Distance of centre of balance mass in the planes of the driving wheels from the axle centre = 0.75 m

Angle between the engine cranks = 90°

Angle between the coupling rod crank with adjacent engine crank = 180°

The balance mass required for the reciprocating parts is equally divided between each pair of coupled wheels. Determine

(a) the magnitude and position of the balance mass required to balance 2/3rd of reciprocating and the complete revolving parts.

(b) The hammer blow and

(c) The maximum variation of tractive force when the locomotive speed is 75 km/h.

■ Solution

Leading wheels

Balance mass = 250 + 0.5 × 2/3 × 300 = 350 kg

Take 2 as reference plane with $\theta_3 = 0°$.

From Table 10.3,

$$\sum M_i r_i \cos\theta_i = 75$$

$$\sum M_i r_i \sin\theta_i = 75$$

$$\sum M_i r_i l_i \cos\theta_i = 58.5$$

$$\sum M_i r_i l_i \sin\theta_i = 61.5$$

$$\left[\left(\sum M_i r_i l_i \cos\theta_i\right)^2 + \left(\sum M_i r_i l_i \sin\theta_i\right)^2\right]^{0.5} = M_5 r_5 l_5$$

$$\left[(58.5)^2 + (61.5)^2\right]^{0.5} = 1.2M_5$$

$$M_5 = \frac{84.88}{1.2} = 70.73 \text{ kg}$$

$$\tan\theta_5 = \frac{-\sum M_i r_i l_i \sin\theta_i}{-\sum M_i r_i l_i \cos\theta_i} = \frac{-61.5}{-58.5} = 1.0513$$

$$\theta_5 = 180° + 46.43° = 226.43° \text{ ccw}$$

Table 10.13

Plane	M kg	r m	Mr	θ deg	$Mr\cos\theta$	$Mr\sin\theta$	l	Mrl	Mrl $\cos\theta$	Mrl $\sin\theta$
1	120	0.25	30	180	-30	0	-0.2	-6	6	0
2	M_2	0.75	$0.75M_2$	θ_2	$0.75M_2 \times \cos\theta_2$	$0.75M_2 \times \sin\theta_2$	0	0	0	0
3	350	0.3	105	0	105	0	0.5	52.5	52.5	0
4	350	0.3	105	90	0	105	1.1	115.5	0	115.5
5	M_5	0.75	$0.75M_5$	θ_5	$0.75M_5 \times \cos\theta_5$	$0.75M_5 \times \sin\theta_5$	1.6	$1.2M_5$	$1.2M_5 \times \cos\theta_5$	$1.2M_5 \times \sin\theta_5$
6	120	0.25	30	270	0	-30	1.8	54	0	-54

Since the numerator and denominator are both negative, therefore θ_5 lies in the third quadrant.

$$M_2 r_2 = \left[\left(\sum M_i r_i \cos\theta_i + M_5 r_5 \cos\theta_5\right)^2 + \left(\sum M_i r_i \sin\theta_i + M_5 r_5 \sin\theta_5\right)^2\right]^{0.5}$$

$$0.75M_2 = \left[(75 + 70.73 \times 0.75 \times \cos 226.43°)^2 + (75 + 70.73 \times 0.75 \times \sin 226.43°)\,2\right]^{0.5}$$

$$= \left[(38.43)^2 + (36.56)^2\right]^{0.5} = 53.046$$

$$M_2 = 70.73 \text{ kg}$$

$$\tan\theta_2 = \frac{-\left(\sum M_i r_i \sin\theta_i + M_5 r_5 \sin\theta_5\right)}{-\left(\sum M_i r_i \cos\theta_i + M_5 r_5 \cos\theta_5\right)}$$

$$= \frac{-36.56}{-38.43} = 0.95134$$

$$\theta_2 = 180° + 43.57° = 223.57°$$

Since the numerator and denominator are both negative, therefore θ_2 lies in the third quadrant.

Trailing wheels Balance mass $= 0.5 \times 2 \times \frac{300}{3} = 100$ kg

Take 2 as reference plane with $\theta_3 = 0°$.

Table 10.14

Plane	M kg	r m	Mr	θ deg	$Mr\cos\theta$	$Mr\sin\theta$	l	Mrl	$Mrl \times \cos\theta$	$Mrl \times \sin\theta$
1	120	0.25	30	180	−30	0	−0.2	−6	6	0
2	M_2	0.75	$0.75M_2$	θ_2	$0.75M_2 \times \cos\theta_2$	$0.75M_2 \times \sin\theta_2$	0	0	0	0
3	100	0.3	30	0	30	0	0.5	15	15	0
4	100	0.3	30	90	0	30	1.1	33	0	33
5	M_5	0.75	$0.75M_5$	θ_5	$0.75M_5 \times \cos\theta_5$	$0.75M_5 \times\sin\theta_5$	1.6	$1.2M_5$	$1.2M_5 \times \cos\theta_5$	$1.2M_5 \times \sin\theta_5$
6	120	0.25	30	270	0	−30	1.8	54	0	−54

From Table 10.14,

$$\sum M_i r_i \cos\theta_i = 0$$

$$\sum M_i r_i \sin\theta_i = 0$$

$$\sum M_i r_i l_i \cos\theta_i = 21$$

$$\sum M_i r_i l_i \sin\theta_i = -21$$

$$\left[\left(\sum M_i r_i l_i \cos\theta_i\right)^2 + \left(\sum M_i r_i l_i \sin\theta_i\right)^2\right]^{0.5} = M_5 r_5 l_5$$

$$\left[(21)^2 + (-21)^2\right]^{0.5} = 1.2M_5$$

$$M_5 = \frac{29.698}{1.2} = 24.75 \text{ kg}$$

$$\tan\theta_5 = \frac{-\sum M_i r_i l_i \sin\theta_i}{-\sum M_i r_i l_i \cos\theta_i} = \frac{-(-21)}{-(21)} = -1$$

$$\theta_5 = 180° - 45° = 135 \text{ ccw}$$

Since the numerator is positive and denominator is negative, therefore θ_5 lies in the second quadrant.

By symmetry,

$$M_2 = 24.75 \text{ kg}$$

$$\tan\theta_2 = \frac{-(21)}{-(-21)} = -1$$

$$\theta_2 = 360° - 45° = 315°$$

Since the numerator is negative and denominator is positive, therefore θ_2 lies in the fourth quadrant.

(b)

$$v = 75 \times \frac{1000}{3600} = 20.83 \text{ m/s}$$

$$\omega = v/r = \frac{20.83}{0.95} = 21.93 \text{ rad/s}$$

Neglecting M_1 and M_6, we have

$$1.2M_5 = \left[15^2 + 33^2\right]^{0.5} = 36.249$$

$$M_5 = 30.2 \text{ kg}$$

Hammer blow $= M_5 b\omega^2 = 30.2 \times 0.75 \times (21.93)^2 = 10893 \text{ N}$

(c) Maximum variation in tractive effort $= \pm\sqrt{2}(1 - c)R\omega^2 r$

$$= \pm\sqrt{2}(1 - 2/3) \times 300 \times (21.93)^2 \times 0.3 = \pm 20404 \text{ N}$$

10.6 MULTI-CYLINDER IN-LINE ENGINES

In a multi-cylinder in-line engine, the cylinder centre lines lie in the same plane and on the same side of the crankshaft centre line, as shown in Fig.10.24.

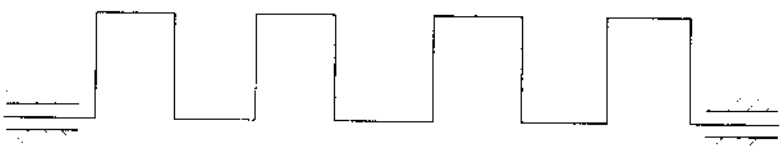


Fig.10.24 Multi-cylinder inline engine

10.6.1 Primary Balancing

The conditions to be satisfied for the primary balancing are:

1. The algebraic sum of the primary forces should be equal to zero, that is, the primary force polygon must close.

$$\sum R\omega^2 r \cos\theta = 0 \tag{10.20a}$$

2. The algebraic sum of the primary couples about any point in the plane of the forces must be equal to zero, that is, the primary couple polygon must close.

$$\sum R\omega^2 r a \cos\theta = 0 \tag{10.20b}$$

where a = distance of the plane of rotation of the crank from a parallel reference plane.

The graphical construction for the balancing of primary forces is represented in Fig.10.25(a) and (b).

ef, fg, gh, eh = primary forces

γ = angle turned through by crankshaft, clockwise

= angle turned through by line of stroke, ccw, i.e. *PQ* goes to *PS*.

kl, ml, mn, nk_1 = primary forces whose resultant is kk_1.

For balance of primary forces d must coincide with o. In a similar way, the primary couples can only be balanced if the couple polygon for the corresponding centrifugal forces is closed.

Hence, if a system of reciprocating masses is to be in primary balance, the system of reciprocating masses, which is obtained by substituting an equal revolving mass at the crankpin for each reciprocating mass, must be balanced.

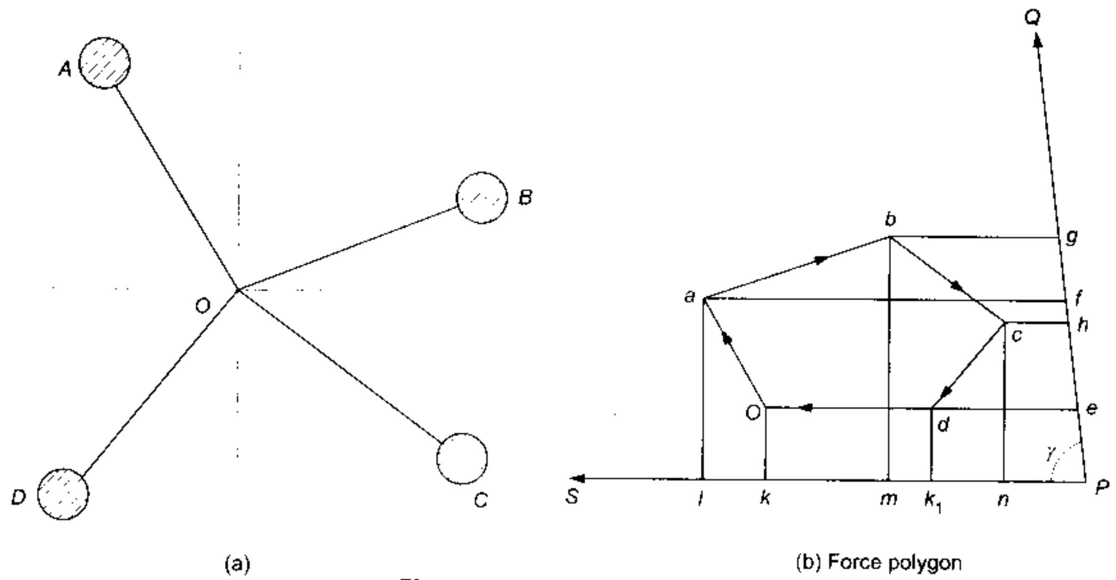


Fig.10.25 Graphical method

10.6.2 Secondary Balancing

The conditions to be satisfied for the secondary balancing are:

1. The algebraic sum of the secondary forces should be equal to zero, that is the secondary force polygon must close.

$$\sum R(2\omega)^2 \left(\frac{r}{4n}\right) \cos 2\theta = 0 \tag{10.21a}$$

2. The algebraic sum of the secondary couples about any point in the plane of the forces must be equal to zero, that is the secondary couple polygon must close.

$$\sum R(2\omega)^2 \left(\frac{r}{4n}\right) a \cos 2\theta = 0 \tag{10.21b}$$

where a is the distance of the plane of rotation of the crank from a parallel reference plane.

$$\text{Imaginary crank length} = \frac{r}{4n} \tag{10.21c}$$

$$\text{Speed} = 2\omega \tag{10.21d}$$

Angle made by imaginary secondary crank with inner dead centre always $= 2\theta$. The actual and imaginary cranks are shown in Fig.10.26.

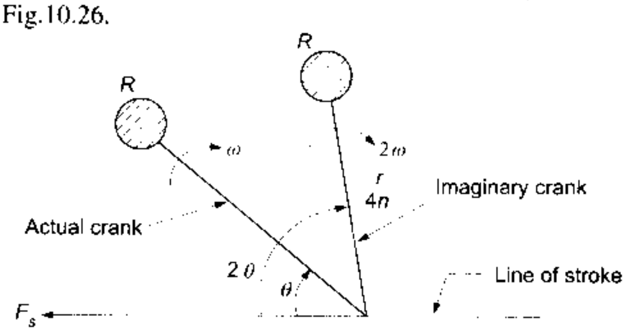


Fig.10.26 Actual and imaginary cranks

Example 10.11

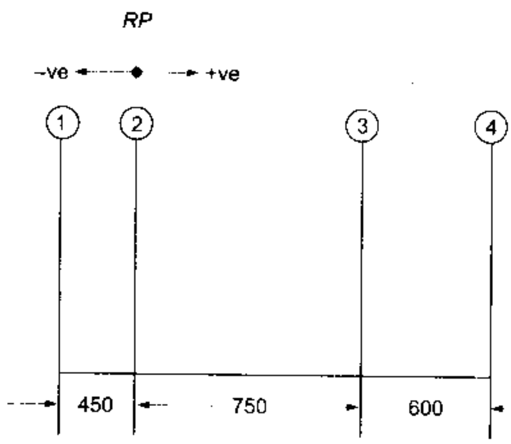
A four crank engine has two outer cranks set at 120° to each other, and their reciprocating masses are each 400 kg. The distance between the planes of rotation of adjacent cranks are 450 mm, 750 mm, and 600 mm. If the engine is to be in complete primary balance, find the reciprocating mass and the relative angular position for each of the inner cranks.

If the length of each crank is 300 mm, length of each connecting rod is 1.2 m and the speed of rotation is 240 rpm, what is the maximum secondary unbalanced force?

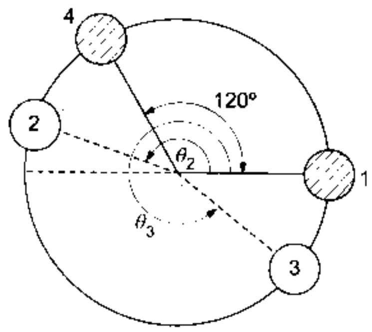
■ Solution

Reciprocating masses:

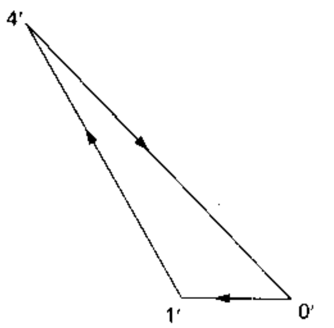
Let M_2 and M_3 be the reciprocating masses for inner cranks 2 and 3; θ_2 and θ_3 their angular locations respectively. The position of planes and primary crank positions is shown in Fig.10.27(a) and (b) respectively.



(a) Position of planes

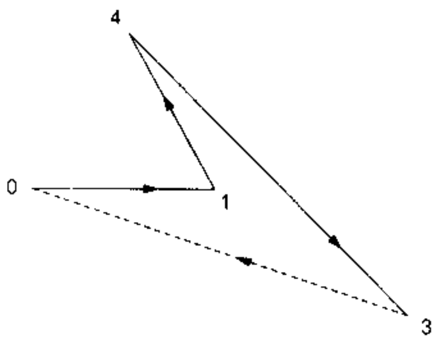


(b) Primary crank positions



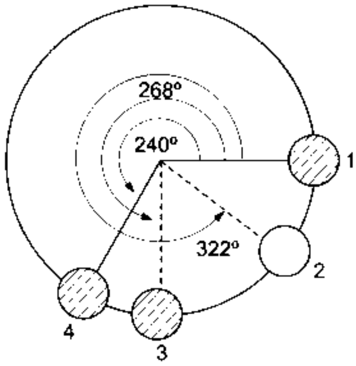
scale: 1 cm = 40 kgm^2

(c) Primary couple polygon

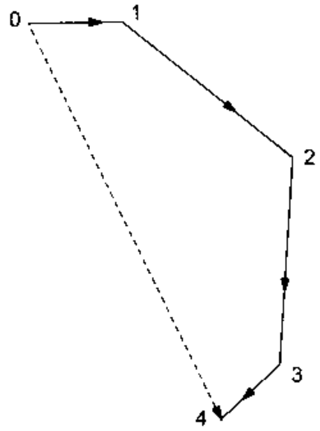


scale: 1 cm = 150 kgm

(d) Primary force polygon



(e) Secondary crank positions



scale: 1 cm = 40 kgm

(f) Secondary force diagram

Fig.10.27

Table 10.15

Plane	Mass M, kg	Radius r, m	$M \cdot r$ kgm	Distance from plane 2 l, m	Couple Mrl, kgm^2
1	400	0.3	120	−0.45	−54.0
2	M_2	0.3	$0.3M_2$	0	0
3	M_3	0.3	$0.3M_3$	0.75	$0.225M_3$
4	400	0.3	120	1.35	162.0

since the engine is to be in complete primary balance, therefore, the primary couple and force polygons must close. The primary couple polygon is shown in Fig.10.27(c), drawn from the data in column 6 of Table 10.15.

$$0.225M_3 = \text{vector } 0'4' = 4.9 \text{ cm} = 196$$

or $$M_3 = 871 \text{ kg}$$

and $$\theta_3 = 314°$$

The force polygon is drawn in Fig.10.24(d), from the data in column 4.

$$0.3M_2 = \text{vector } 03 = 284$$

or $$M_2 = 947 \text{ kg}$$

and $$\theta_2 = 168°$$

Secondary unbalanced force The secondary cranks at twice the angle are shown in Fig.10.27(e). The secondary force polygon is drawn in Fig.10.27(f). The closing side of the polygon gives the unbalanced secondary force.

$$\text{Maximum unbalanced secondary force} = \frac{582\omega^2}{n} = 582 \times \frac{\left(2\pi \times \frac{240}{60}\right)^2}{1.2/0.3} = 91.96 \text{ kN}$$

10.6.3 In-line Two Cylinder Engine

Consider line diagram of a two cylinder in-line engine shown in Fig.10.28. The cranks are 180° apart and have equal reciprocating masses. Taking a plane through the centre line as the reference plane, we have

Primary force, $F_p = Rr\omega^2 [\cos\theta + \cos(180° + \theta)] = 0$

Primary couple, $C_p = Rr\omega^2 [0.5a\cos\theta - 0.5a\cos(180° + \theta)]$

$$= Rr\omega^2 a\cos\theta \quad (10.22a)$$

$$(C_p)_{max} = Rr\omega^2 a \quad \text{at} \quad \theta = 0° \text{ and } 180° \quad (10.22b)$$

Secondary force, $F_s = \left(\frac{Rr\omega^2}{n}\right)[\cos 2\theta + \cos 2(180° + \theta)]$

$$= \left(\frac{2Rr\omega^2}{n}\right)\cos 2\theta \quad (10.23a)$$

$$(F_s)_{max} = \frac{2Rr\omega^2}{n} \quad \text{at} \quad \theta = 0°, 90°, 180°, 270° \quad (10.23b)$$

Secondary couple, $C_s = \left(\frac{Rr\omega^2}{n}\right)[0.5a\cos 2\theta - 0.5a\cos 2(180° + \theta)]$

$$= 0$$

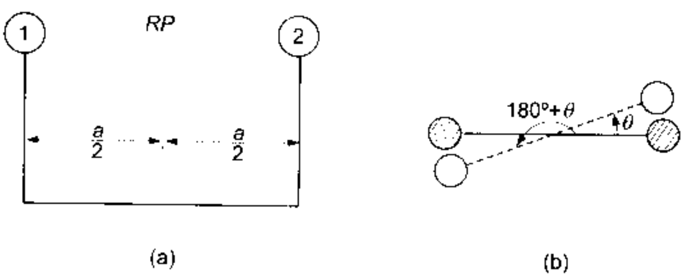


Fig.10.28 In-line two cylinder engine

The force and couple polygons are shown in Fig.10.29(a) to (d).

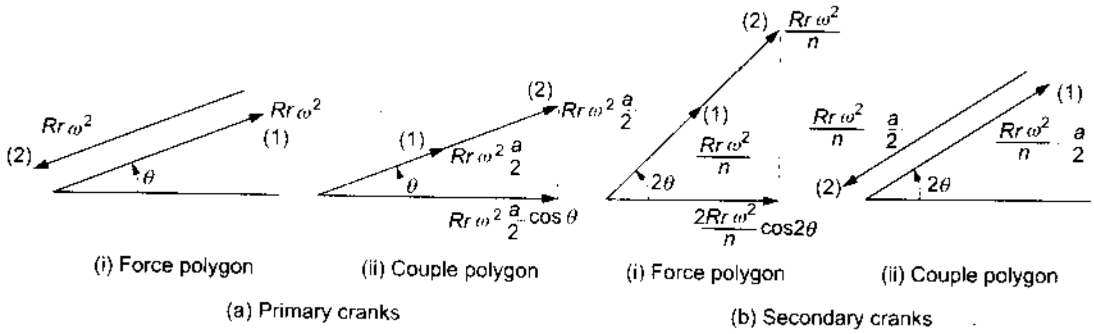


Fig.10.29

10.6.4 In-line Four Cylinder Four-stroke Engine

A line diagram of a four cylinder engine is shown in Fig.10.30. The forces and couples are:

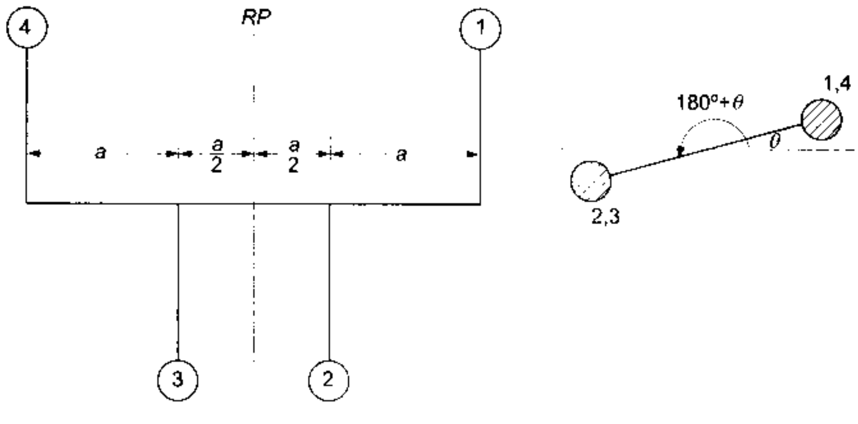


Fig.10.30

Primary force, $F_p = R r \omega^2 [\cos\theta + \cos(180° + \theta) + \cos(180° + \theta) + \cos\theta] = 0$

Primary couple, $C_p = R r \omega^2 [1.5a \cos\theta + 0.5a \cos(180° + \theta) - 0.5a \cos(180° + \theta) - 1.5a \cos\theta] = 0$

Secondary force, $F_s = \left(\frac{R r \omega^2}{n}\right) [\cos 2\theta + \cos 2(180° + \theta) + \cos 2(180° + \theta) + \cos 2\theta]$

$$= \left(\frac{4 R r \omega^2}{n}\right) \cos 2\theta \qquad (10.25a)$$

$$(F_s)_{max} = 4Rr\frac{\omega^2}{n} \quad \text{at} \quad \theta = 0°, 90°, 180°, 270° \qquad (10.25b)$$

Secondary couple, $C_s = \left(Rr\frac{\omega^2}{n}\right) [1.5a \cos 2\theta + 0.5a \cos 2(180° + \theta) - 0.5a \cos 2(180° + \theta) - 1.5a \cos 2\theta] = 0$

The force and couple polygons are shown in Fig.10.31(a) to (d).

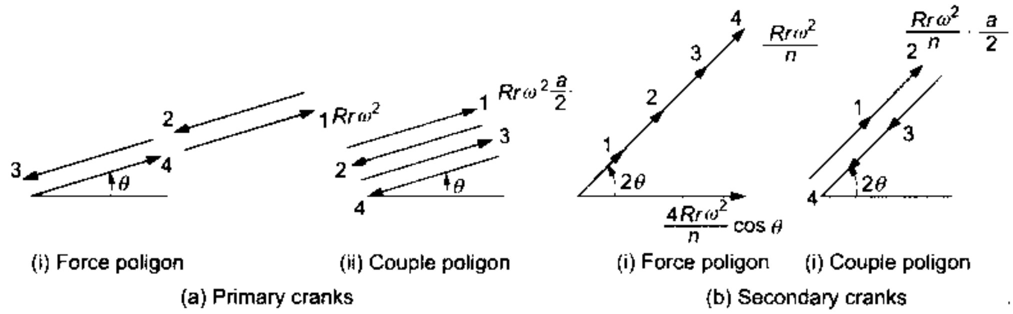


Fig.10.31

Example 10.12

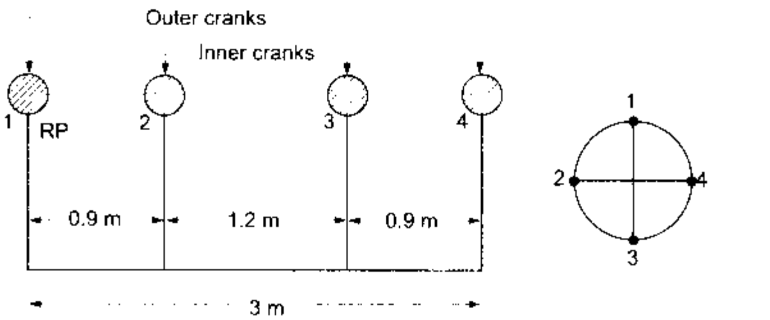
In a marine oil engine, the cranks of four cylinders are arranged at angular displacements of 90°. The speed of the engine is 105 rpm and the mass of reciprocating parts for each cylinder is 850 kg. Each crank is 0.4 m long. The outer cranks are 3 m apart and the inner cranks are 1.2 m apart and are placed symmetrically between the outer cranks.

Find the firing order of the cylinders for the best primary balancing of reciprocating parts and also the maximum unbalanced primary couple for this arrangement.

■ Solution

$$\omega = 2\pi \times \frac{105}{60} = 10.996 \text{ rad/s}$$
$$r = 0.4 \text{ m}$$

The primary forces are always balanced as cranks are arranged at an angular displacement of 90° to each other. The primary couples need to be investigated. The position of cranks is as shown in Fig.10.32.



(a) Crank position (b) Crank arrangement

Fig.10.32

The possible firing orders are: 1234, 1243, 1423, 1324, 1342, 1432, as shown in Table 10.17.

The disturbing force along the axis of the cylinder = $Mr\omega^2 \cos\theta$

Let $K = Mr\omega^2 = 850 \times 0.4 \times (10.996)^2 = 41107$ N, as shown in Table 10.16.

Total disturbing force

$$= \sum_{l=1}^{4} K \cos(\theta + \alpha_l)$$

where α_l = angle between the reference crank and the crank considered.

Table 10.16

Plane of cylinder	M	Mr	$K = Mr\omega^2$	Arm length l	Couple Kl
1	850	340	41107	0	0
2	850	340	41107	0.9	$0.9K$
3	850	340	41107	2.1	$2.1K$
4	850	340	41107	3	$3K$

Table 10.17

Disposition of cranks	Crank positions	Primary couple polygon	Resultant primary couple
1234			$\left[(3-0.9)^2+(2.1)^2\right]^{0.5}\times K = 2.97K$
1243			$\left[(2.1-0.9)^2+(3)^2\right]^{0.5}\times K = 3.231K$
1423			$\left[(3-2.1)^2+(0.9)^2\right]^{0.5}\times K = 1.273K$
1324			$\left[(3-2.1)^2+(0.9)^2\right]^{0.5}\times K = 1.273K$
1342			$\left[(2.1-0.9)^2+(3)^2\right]^{0.5}\times K = 3.231K$
1432			$\left[(3-0.9)^2+(2.1)^2\right]^{0.5}\times K = 2.97K$

$$\text{Least value of primary couple} = 1.273K = 1.273\times 41107 = 52329\text{ Nm}$$

Best firing order is 1423 and 1324.

10.7 BALANCING OF RADIAL ENGINES

10.7.1 Direct and Reverse Cranks Method

This method is used to balance radial or V-engines, in which connecting rods are connected to a common crank, as shown in Fig.10.33. Since the plane of various cranks is the same, therefore, there is no unbalanced primary or secondary couple.

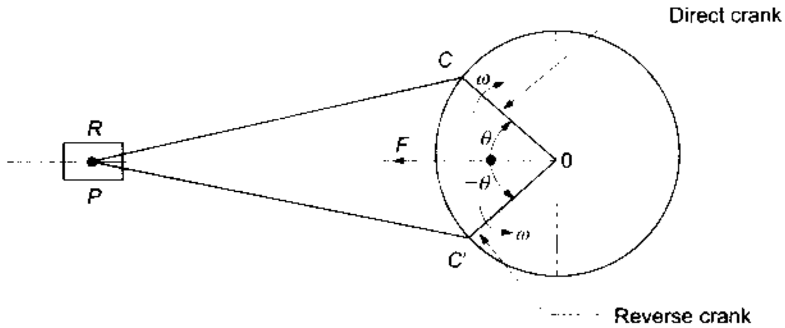


Fig.10.33 Direct and reverse crank method.

Let the direct crank OC rotate uniformly at ω rad/s speed in a clockwise direction. Then the reverse crank OC ′ will rotate in the ccw direction. The inverse crank OC ′ is the mirror image of the direct crank OC.

Primary forces

Now the primary force, $F_p = R\omega^2 r \cos\theta$

This force is equal to the component of the centrifugal force along the line of stroke, produced by a mass R placed at the crankpin C. Let us suppose that the mass R is divided into two equal parts, each equal to $R/2$. It is assumed that $R/2$ is fixed at the direct crankpin C and the other half $R/2$ is fixed at the reverse crankpin C ′.

Centrifugal force acting on the primary direct and reverse crankpins $= 0.5R\omega^2 r$

Component of the centrifugal force on the direct crank acting along the line of stroke from O to P,

$$F_{pd} = 0.5R\omega^2 r \cos\theta$$

Component of the centrifugal force on the reverse crank acting along the line of stroke from O to P

$$F_{pr} = 0.5R\omega^2 r \cos\theta$$

Total component of the centrifugal force along the line of stroke

$$F_p = F_{pd} + F_{pr} = R\omega^2 r \cos\theta \tag{10.26}$$

Which is the primary force itself. Hence, for primary force effects, the mass of the reciprocating parts at P may be replaced by two masses at crankpins C and C ′, each of mass $R/2$ at radii equal to r.

Secondary forces

The secondary force, $$F_s = R\omega^2 r \frac{\cos 2\theta}{n} \tag{10.27}$$

In the similar way as discussed for the primary force, the secondary force effect may be taken into account by dividing the mass R into two equal parts and placing it at the imaginary crankpins at radii $\frac{r}{4n}$.

Example 10.13

The three cylinders of an air compressor have their axes 120° to one another and their connecting rods are coupled to a common crank. The stroke is 100 mm and the length of each connecting rod is 150 mm. The mass of the reciprocating parts per cylinder is 2 kg. Find the maximum primary and secondary forces acting on the frame of the compressor when running at 3000 rpm.

■ **Solution**

The position of three cylinders is shown in Fig.10.34(a), with the common crank along the inner dead centre of cylinder 1.

Primary forces

The primary direct and reverse crank positions are shown in Fig.10.34(b).

1. Since $\theta = 0°$ for cylinder 1, both the primary direct and reverse cranks will coincide with the common crank.
2. Since $\theta = \pm 120°$ for cylinder 2, the primary direct crank is 120° clockwise and the primary reverse crank is 120° counter-clockwise from the line of stroke of cylinder 2.
3. Since $\theta = \pm 240°$ for cylinder 3, the primary direct crank is 240° clockwise and primary reverse crank is 240° counter-clockwise from the line of stroke of cylinder 3.

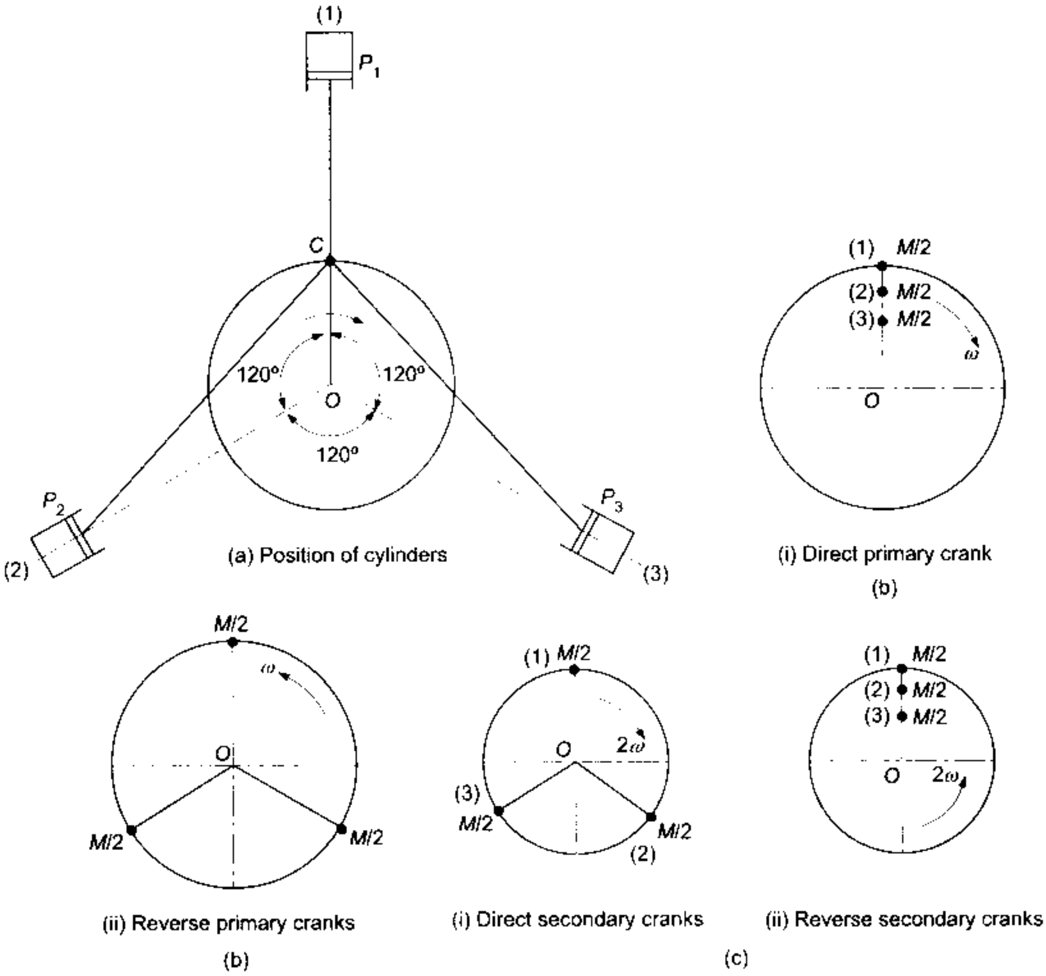


Fig.10.34 Direct and reverse crank method

From Fig.10.34 (b-ii), we find that the primary reverse cranks form a balanced system. Therefore, there is no unbalanced primary force due to the reverse cranks. From Fig.10.34 (b-i), we find that the resultant primary force is equivalent to the centrifugal force of a mass 1.5 M attached to the end of the crank.

$$\text{Maximum primary force} = 1.5M\omega^2 r$$

$$= 1.5 \times 2\left(2\pi \times \frac{3000}{60}\right)^2 0.05$$

$$= 14804.4 \text{ N}$$

The maximum primary force may be balanced by a mass attached diametrically opposite to the crank pin and rotating with the crank, of magnitude B_1 at radius b_1, such that

$$B_1 b_1 = 1.5Mr = 1.5 \times 2 \times 0.05 = 0.15 \text{ Nm}$$

Secondary force

The secondary direct and reverse crank positions are shown in Fig.10.34(c).

1. Since $2\theta = 0°$ for cylinder 1, both the secondary direct and reverse cranks will coincide with the common crank.
2. Since $2\theta = \pm 240°$ for cylinder 2, the secondary direct crank is 240° clockwise and the secondary reverse crank is 240° counter-clockwise from the line of stroke of cylinder 2.
3. Since $2\theta = \pm 480°$ for cylinder 3, the secondary direct crank is 480° or 120° clockwise and secondary reverse crank is 480° or 120° counter-clockwise from the line of stroke of cylinder 3.

The resultant secondary force = 1.5 M attached to a crank at radius $r/4n$ rotating at 2ω speed.

$$\text{Maximum secondary force} = 1.5M(2\omega)^2\left(\frac{r}{4n}\right)$$
$$= 1.5 \times 2\left(4\pi \times \frac{3000}{60}\right)^2\left(\frac{0.05}{4 \times 3}\right)$$
$$= 4934.8 \text{ N}$$

The maximum secondary force can be balanced by a mass B_2 at radius b_2 attached diametrically opposite to the crank pin, and rotating ccw at twice the speed, such that

$$B_2 b_2 = 1.5M \times \frac{r}{4n}$$
$$= 1.5 \times 2 \times \frac{0.05}{4 \times 3} = 0.0125 \text{ Nm}$$

10.8 BALANCING OF V-ENGINES

Consider a symmetrical two cylinder V-engine, as shown in Fig.10.35. The common crank OC is driven by two connecting rods PC and QC. The lines of stroke OP and OQ are inclined to the vertical OY, at an angle α.

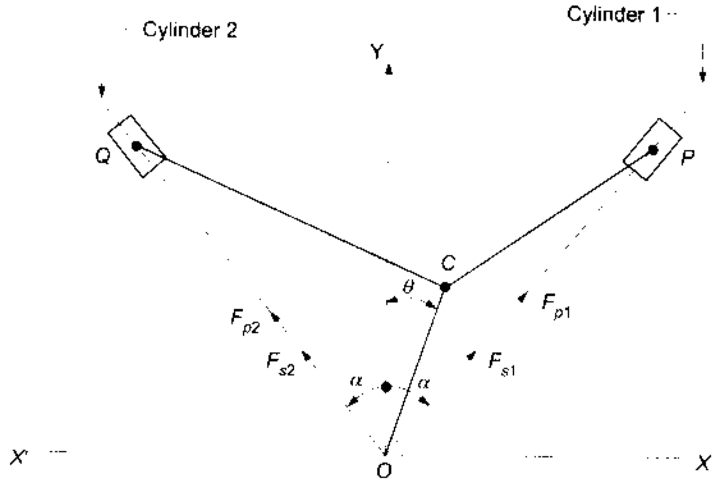


Fig.10.35 V-engine

Inertia force due to the reciprocating parts of cylinder 1, along the line of stroke

$$= R\omega^2 r\left[\cos(\alpha - \theta) + \frac{\cos 2(\alpha - \theta)}{n}\right]$$

Inertia force due to the reciprocating parts of cylinder 2, along the line of stroke

$$= R\omega^2 r\left[\cos(\alpha+\theta) + \frac{\cos 2(\alpha+\theta)}{n}\right]$$

The plane of cranks is the same, therefore, there are no primary or secondary couples.

Primary forces Primary force of cylinder 1 acting along the line of stroke

$$F_{p1} = R\omega^2 r\cos(\alpha-\theta)$$

Component of F_{p1} along the vertical line OY $= F_{p1}\cos\alpha$

$= R\omega^2 r\cos(\alpha-\theta)\cos\alpha$

Component of F_{p1} along the horizontal line OX $= F_{p1}\sin\alpha$

$= R\omega^2 r\cos(\alpha-\theta)\sin\alpha$

Similarly, for the cylinder 2, we have $F_{p2} = R\omega^2 r\cos(\alpha+\theta)$

Component of F_{p2} along the vertical line OY $= F_{p2}\cos\alpha$

$= R\omega^2 r\cos(\alpha+\theta)\cos\alpha$

Component of F_{p2} along the horizontal line OX' $= F_{p2}\sin\alpha$

$= R\omega^2 r\cos(\alpha+\theta)\sin\alpha$

Total component of primary force along the vertical line OY

$$\begin{aligned} F_{pv} &= R\omega^2 r[\cos(\alpha-\theta)+\cos(\alpha+\theta)]\cos\alpha \\ &= 2R\omega^2 r\cos^2\alpha\cos\theta \end{aligned}$$

Total component of primary force along the horizontal line OX

$$\begin{aligned} F_{ph} &= R\omega^2 r[\cos(\alpha-\theta)-\cos(\alpha+\theta)]\sin\alpha \\ &= 2R\omega^2 r\sin^2\alpha\sin\theta \end{aligned}$$

Resultant primary force,

$$\begin{aligned} F_p &= \left[F_{pv}^2 + F_{ph}^2\right]^{0.5} \\ &= 2R\omega^2 r\left[(\cos^2\alpha\cos\theta)^2 + (\sin^2\alpha\sin\theta)^2\right]^{0.5} \end{aligned} \quad (10.28)$$

For $\alpha = 45°$, we have

$$F_p = R\omega^2 r \quad (10.29)$$

Secondary forces Secondary force of cylinder 1 acting along the line of stroke

$$F_{s1} = R\omega^2 r\frac{\cos 2(\alpha-\theta)}{n}$$

Component of F_{s1} along the vertical line OY $= F_{s1}\cos\alpha$

$$= R\omega^2 r \cdot \frac{\cos 2(\alpha - \theta)}{n} \cdot \cos\alpha$$

Component of F_{s1} along the horizontal line OX $= F_{s1}\sin\alpha$

$$= R\omega^2 r \cdot \frac{\cos 2(\alpha - \theta)}{n} \cdot \sin\alpha$$

Similarly, for the cylinder 2, we have

$$F_{s2} = R\omega^2 r \frac{\cos 2(\alpha + \theta)}{n}$$

Component of F_{s2} along the vertical line OY $= F_{s2}\cos\alpha$

$$= R\omega^2 r \cdot \frac{\cos 2(\alpha + \theta)}{n} \cdot \cos\alpha$$

Component of F_{s2} along the horizontal line OX' $= F_{s2}\sin\alpha$

$$= R\omega^2 r \frac{\cos 2(\alpha + \theta)}{n} \cdot \sin\alpha$$

Total component of secondary force along the vertical line OY

$$F_{sv} = R\omega^2 r[\cos 2(\alpha - \theta) + \cos 2(\alpha + \theta)]\frac{\cos\alpha}{n}$$

$$= 2R\omega^2 \left(\frac{r}{n}\right)\cos\alpha\cos 2\alpha\cos 2\theta$$

Total component of secondary force along the horizontal line OX

$$F_{sh} = R\omega^2 r[\cos 2(\alpha - \theta) - \cos 2(\alpha + \theta)]\frac{\sin\alpha}{n}$$

$$= 2R\omega^2 \left(\frac{r}{n}\right)\sin\alpha\sin 2\alpha\sin 2\theta$$

Resultant secondary force, $F_s = \left[F_{sv}^2 + F_{sh}^2\right]^{0.5}$

$$= 2R\omega^2 \left(\frac{r}{n}\right)\left[(\cos\alpha\cos 2\alpha\cos 2\theta)^2 + (\sin\alpha\sin 2\alpha\sin 2\theta)^2\right]^{0.5} \quad (10.30)$$

For $\alpha = 45°$, we have

$$F_s = \left(\frac{\sqrt{3}}{2}\right) R\omega^2 \left(\frac{r}{n}\right) \quad (10.31)$$

Example 10.14

A V-twin engine has the cylinder axes at right angles and the connecting rods operate a common crank. The reciprocating mass per cylinder is 10 kg and the crank radius is 80 mm. The length of connecting rod is 0.4 m. Show that the engine may be balanced for primary forces by means of a revolving balance mass.

If the engine speed is 600 rpm, what is the value of maximum resultant secondary force?

■ Solution

Here $\alpha = 45°,\ n = \dfrac{0.4}{0.08} = 5$

and $\omega = 2\pi \times \dfrac{600}{60} = 62.83$ rad/s

Resultant primary force, $F_p = 2R\omega^2 r\left[(\cos^2\alpha\cos\theta)^2 + (\sin^2\alpha\sin\theta)^2\right]^{0.5}$

Since the resultant primary force $R\omega^2 r$ is the centrifugal force of a mass R at the crank pin radius rotating at speed ω, the engine may be balanced by a rotating balance mass.

Maximum resultant secondary force,

$$F_s = 2R\omega^2\left(\frac{r}{n}\right)\left[(\cos\alpha\cos 2\alpha\cos 2\theta)^2 + (\sin\alpha\sin 2\alpha\sin 2\theta)^2\right]^{0.5}$$

For $\alpha = 45°$, $F_s = \sqrt{2}R\omega^2\left(\frac{r}{n}\right)\sin 2\theta$

For a maximum value $\sin 2\theta = \pm 1$, or $\theta = 45°$ and $135°$.

Maximum resultant secondary force,

$$(F_s)_{\max} = \sqrt{2}\left(R\frac{r}{n}\right)\omega^2 = \sqrt{2}\left(10\times\frac{0.08}{5}\right)(62.83)^2 = 893.24 \text{ N}$$

Exercises

1 A rotating shaft carries four radial masses $A = 8$ kg, B, $C = 6$ kg, and $D = 5$ kg. The mass centres are 30 mm, 40 mm, 40 mm and 50 mm respectively from the axis of the shaft. The axial distance between the planes of rotation of A and B is 400 mm and between B and C is 500 mm. The masses A and C are at right angles to each other. Find for a complete balance, (a) the angle of the masses B and D from mass A, (b) the axial distance between the planes of rotation of C and D and (c) the magnitude of mass B.

2 A rotating shaft carries four unbalanced masses 20 kg, 15 kg, 18 kg and 12 kg at radii 50 mm, 60 mm, 70 mm and 60 mm respectively. The second, third and fourth masses revolve in planes 100 mm, 150 mm and 300 mm respectively measured from the plane of first mass and at angular locations of 60°, 120° and 280° respectively measured clockwise from the first mass. The shaft is dynamically balanced by two masses, both located at 50 mm radii and revolving in planes midway between those of first and second masses and midway between those of third and fourth masses. Determine graphically the magnitudes of the masses and their angular positions.

3 A shaft of span 3 m between two bearings carries two masses of 15 kg and 30 kg acting at the extremities of the arms 0.5 m and 0.6 m respectively. The planes in which these masses rotate are 1 m and 2 m respectively from the left end bearing. The angle between the arms is 60°. The speed of rotation of the shaft is 240 rpm. If the masses are balanced by two counter masses rotating with the shaft acting at radii of 0.25 m and placed at 0.3 m from each bearing centre, determine the magnitude of the two balance masses and their orientation with respect to the 15 kg mass.

4 A 4 m long shaft carries three pulleys, two at its ends and the third at the midpoint. The two end pulleys has mass of 80 kg and 40 kg and their centre of gravity are 3 mm and 5 mm respectively from the axis of the shaft. The middle pulley mass is 50 kg and its centre of gravity is 8 mm from the shaft axis. The pulleys are keyed to the shaft and the assembly is in static balance. The shaft rotates at 300 rpm in two bearings 2.5 m apart with equal overhang on either side. Determine (a) the relative angular positions of the pulleys, and (b) dynamic reactions at the two bearings.

5 A single cylinder horizontal engine runs at 120 rpm with a stroke of 400 mm. The mass of the revolving parts assumed concentrated at the crankpin is 100 kg and mass of the reciprocating parts is 150 kg. Determine the magnitude of the balancing mass to be placed opposite to the crank at a radius of 150 mm

which is equivalent to all the revolving and 2/3rd of the reciprocating parts. If the crank turns 30° from the inner dead centre, find the magnitude of the unbalanced force due to the balancing mass.

6 A single cylinder engine runs at 240 rpm and has a stroke of 200 mm. The reciprocating parts has a mass of 120 kg and the revolving parts are equivalent to a mass of 80 kg at a radius of 100 mm. A mass is placed opposite to the crank at a radius of 150 mm to balance the whole of the revolving mass and 2/3rd of the reciprocating mass. Determine the magnitude of the balancing mass and the resultant residual unbalance force when the crank has turned 30° from the inner dead centre. Neglect the obliquity of the connecting rod.

7 A two-cylinder uncoupled locomotive with cranks at 90° has a crank radius of 320 mm. The distance between the centres of driving wheels is 1.5 m. The pitch of cylinders is 0.6 m. The diameter of treads of driving wheels is 1.8 m. The radius of centres of gravity of balance masses is 0.7 m. The pressure due to dead load on each of the wheels is 40 kN. The masses of the reciprocating and rotating parts per cylinder are 300 kg and 350 kg respectively. The speed of the locomotive is 60 km/h. Find (a) the balancing masses in magnitude and position in the planes of driving wheels to balance whole of the revolving and 2/3rd of the reciprocating parts, (b) the swaying couple, (c) the variation in tractive effort, (d) the maximum and minimum pressure on the rails and (e) the maximum speed at which it is possible to run the locomotive, in order that the wheels are not lifted from the rails.

8 A four cylinder engine has two outer cranks at 120° to each other and their reciprocating masses are each 400 kg. The distance between the planes of rotation of the adjacent cranks are 0.4 m, 0.7 m, 0.7 m and 0.5 m. Find the reciprocating mass and the relative angular position for each of the inner cranks, if the engine is to be in complete primary balance. Also find the maximum secondary force, if the length of each crank is 0.4 m, the length of each connecting rod 1.8 m and the engine speed 480 rpm.

9 In a four crank symmetrical engine, the reciprocating masses of the two outside cylinders A and D are each 600 kg and those of the two inside cylinders B and C are each 900 kg. The distance between the cylinder axes of A and D is 5 m. Taking the reference line to bisect the angle between the cranks A and D and the reference plane to bisect the distance between the cylinder axes of A and D, find the angles between the cranks and the distance between the cylinder axes of B and C for complete balance except for secondary couples.

Determine the maximum value of the unbalanced secondary couple if the length of the crank is 0.4 m, length of the connecting rod 1.8 m and speed is 180 rpm.

10 A three cylinder radial engine driven by a common crank has the cylinders spaced at 120°. The stroke is 120 mm, the length of connecting rod 240 mm, the mass of the reciprocating parts per cylinder is 1 kg and the speed of the crank shaft is 2400 rpm. Determine the magnitude of the primary and secondary forces.

11 A two cylinder V-engine has the cylinders set at an angle of 45°, with both pistons connected to the single crank. The crank radius is 60 mm and the connecting rods are 300 mm long. The reciprocating mass per line is 1.5 kg and the total rotating mass is equivalent to 2 kg at the crank radius. A balance mass fitted opposite to the crank is equivalent to 2.5 kg at a radius of 90 mm. Determine for an engine speed of the maximum and minimum values of the primary and secondary forces due to the inertia of reciprocating and rotating masses, for engine speed of 1800 rpm.

12 Explain the following:

(a) Balancing of multi-cylinder in-line engines

(b) Partial balancing of two cylinder steam locomotive

(c) Discuss the effect of primary, secondary forces and couples in a four-cylinder in-line four stroke engine with mathematical analysis. Compare the above engine with three-cylinder in-line engine in respect of balancing.

(d) What is the effect of partial primary balance of a reciprocating engine?

13 A two cylinder locomotive with cranks at 90° has a crank radius of 325 mm. The distance between centres of driving wheels is 1.5 m. The pitch of cylinders is 600 mm. The diameter of treads of driving wheels is 1.8 m. The radius of centres of gravity of balance weights is 650 mm. The pressure due to dead load on each wheel is 40 kN. The weights of reciprocating and rotating parts per cylinder are 3.3 kN and 3 kN respectively. The speed of the locomotive is 60 km/h. Find

(a) the balancing weights both in magnitude and position required to be placed in the planes of driving wheels to balance whole of the revolving and two-third of the reciprocating masses,

(b) the swaying couple,

(c) the variation of tractive effort,

(d) the maximum and minimum pressure on rails and

(e) the maximum speed at which it is possible to run the locomotive, in order that the wheels are not lifted from the rails.

14 In a four-cylinder petrol engine equally spaced, the cranks, numbered from end are 1, 2, 3 and 4. Cranks 1 and 4 are in phase and 180° ahead of cranks 2 and 3. The reciprocating masses of each cylinder weigh 10 N. The cranks are 50 mm radius and the connecting rods 200 mm long.

What are the resultant unbalanced forces and couples, primary and secondary, when cranks 1 and 4 are on top dead centre position? The engine is rotating at 1500 rpm in a clockwise direction when viewed from the front. Take the reference plane midway between the cylinders 2 and 3.

15 Four weights *A*, *B*, *C* and *D* revolve at equal radii and are equally spaced along shaft. The weights weigh 70 N and the radii of *C* and *D* make angles of 90° and 240° respectively with the radius of *B*. Find the magnitude of the weights *A*, *C* and *D* and the angular position of *A* so that the system may be completely balanced.

16 Explain the terms (a) variation in tractive effort, (b) swaying couple and (c) hammer blow as applied to locomotive balancing. Derive expressions for these for a two-cylinder locomotive having cranks 90° apart.

17 A twin cylinder uncoupled locomotive has its cylinders 0.6 m apart and balance weights are 60° apart. The planes being symmetrically placed about the centre line. For each cylinder the revolving and reciprocating masses are 300 kg and 285 kg at the crank pin radius of 320 mm. All the revolving and 2/3rd of the reciprocating masses are balanced. The driving wheels are 1.8 m diameter. When the engine runs at 60 km/h, find (a) the swaying couple, (b) the variation in tractive effort, and (c) the hammer blow.

18 Investigate the state of primary and secondary balancing of a four-stroke cycle four-cylinder engine with a firing order of I-II-III-IV. What will be the change in this state when the firing is altered to I-II-IV-III?

19 For a V-engine consisting of two cylinders spaced apart by an angle α, show that the engine may be passively for first-order forces, but the second-order horizontal forces remain unbalanced.

20 Obtain the expressions for primary and secondary forces for a V-engine having two identical cylinders lying in a plane. The included angle between the cylinder centre lines is 2α.

21 Explain clearly the difference in the nature of unbalance caused by primary and secondary disturbing forces in the case of a reciprocating mass. What is the essential difference between unbalance caused by a reciprocating mass and that caused by a revolving mass? How will you achieve the complete balance in the case of a multi-cylinder in-line engine?

22 The reciprocating mass per cylinder in a 60° V-engine is 1.2 kg. The stroke and the connecting rod length are 100 mm and 250 mm respectively. If the engine runs at 2000 rpm, determine the maximum and minimum values of the primary and secondary forces. Also find out the crank positions corresponding to these values.

23 The following data refer to a two-cylinder locomotive with cranks at 90°:

Reciprocating mass per cylinder = 300 kg

Crank radius = 300 mm

Diameter of the driving wheels = 1.8 m

Distance between the cylinder centre lines = 0.65 m

Distance between the driving wheel centre planes = 1.55 m

Determine (a) the fraction of the reciprocating masses to be balanced by placing the balancing masses on the driving wheels, if the hammer blow is not to exceed 46 kN at 96.5 km/h and (b) the variation in tractive effort.

24 (a) Why do high speed rotating and reciprocating machinery need to be balanced in practice?

(b) A shaft with 3 m span between two bearings carries two weights of 100 N and 200 N acting at the extremities of arms 0.45 m and 0.60 m long respectively. The planes in which these weights rotate are 1.2 m and 2.4 m respectively from the left end bearing supporting the shaft (Fig.10.36). The angle between these arms is 60° as indicated in the inset (a) of Fig.10.36 The speed of rotation of the shaft is 200 rpm. If the weights are balanced by two counter-weights rotating with the shaft acting at radii of 0.3 m and placed at 0.3 m from each bearing centre, estimate the magnitude of the two balance weights and their orientation with respect to the x-axis, that is, load A.

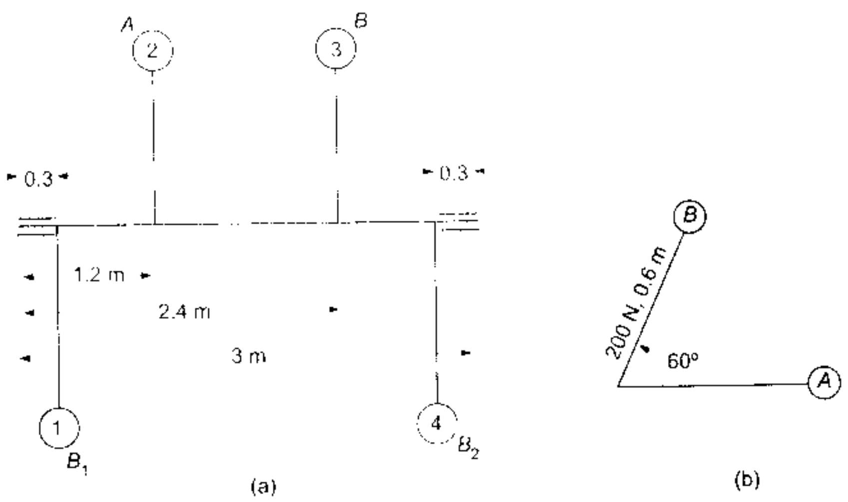


Fig.10.36